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On edge perfectness and classes of bipartite graphs

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Abstract

We define a notion of dependence for the edges of a graph and derive a concept of edge perfectness. We give some examples of classes of bipartite edge-perfect graphs. Moreover the complexity of the corresponding algorithms is investigated.

1. Introduction

Throughout this paper all graphs are finite, simple and undirected. Formally a *graph* is a pair $G = (V,E)$ of a finite set V (of *vertices*) and a set $E \subseteq \binom{V}{2}$ (of *edges*). We denote by $V(G)$ the vertex set V of $G = (V,E)$ and by $E(G)$ the edge set E of G. Two graphs G and H are *isomorphic* if there is a bijection between $V(G)$ and $V(H)$ respecting the edge relations. In what follows, we will not distinguish between isomorphic graphs.

For a vertex v we denote its (open) neighbourhood (in the graph G) by $N(v) = \{w : \{v,w\} \in E(G)\}$ $(= N_G(v))$, this is the set of vertices *adjacent* to v (in G). The closed neighbourhood $N[v]$ is defined by $N[v] = N(v) \cup \{v\}$. By $I(v)$ we denote the set of edges *incident* with v, $I(v) = \{e : e \in E \text{ and } v \in e\}$. For a set $F \subseteq E$ we denote by $\bigcup F$ the set of vertices incident with at least one edge of F.

A graph $H = (W,F)$ is an *induced subgraph* of $G = (V,E)$ if $W \subseteq V$ and $F = E \cap \binom{W}{2}$. In this case we refer to H by $G[W]$. For $W \subseteq V$ we denote the graph $G[V \setminus W]$ by $G - W$ and $G - v$ denotes $G - \{v\}$ for a vertex v.

A sequence $(v_1, v_2, \ldots, v_k)$ of pairwise distinct vertices forms a *path* in G if, for $i = 2, \ldots, k$, $\{v_{i-1}, v_i\}$ is an edge of G. The length of this path is $k - 1$. The path $(v_1, v_2, \ldots, v_k)$ forms a *cycle* in G if $\{v_1, v_k\}$ is an edge of G. The length of this cycle

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is k. An edge $\{v_i, v_j\}$ is a *chord* of the path $(v_1, \ldots, v_k)$ if $|i-j| > 1$. $\{v_1, v_k\}$ is a chord of the path $(v_1, \ldots, v_k)$, but it is not a chord of the cycle $(v_1, \ldots, v_k)$. An (induced sub)graph isomorphic to a chordless path or cycle with k vertices is denoted by P_k and C_k, respectively. Analogously K_i denotes the complete (sub)graph on i vertices, and $K_{i,j}$ denotes the complete bipartite (sub)graph on colour classes of size i and j.

A graph $G = (V, E)$ is connected if for any pair of distinct vertices u and v in V there exists a path $(u, \ldots, v)$ in G. $S \subset V$ is a *cutset* of G if $G - S$ is not connected.

Definition 1. $C \subseteq V$ is a *clique* in $G = (V, E)$ if every pair of distinct vertices in C is adjacent in G. $I \subseteq V$ is an *independent set* in $G = (V, E)$ if no pair of distinct vertices in I is adjacent in G.

$$\alpha(G) = \max\{|I| : I \text{ is an independent set in } G\},$$

$$\omega(G) = \max\{|C| : C \text{ is a clique in } G\},$$

$$\chi(G) = \min\{k : V \text{ is covered by independent sets } I_1, I_2, \ldots, I_k \text{ in } G\},$$

$$\kappa(G) = \min\{k : V \text{ is covered by cliques } C_1, C_2, \ldots, C_k \text{ in } G\}.$$

The *complement* of a graph $G = (V, E)$ is the graph $G^c = (V, \binom{V}{2} \setminus E)$. A graph $G = (V, E)$ is said to be *bipartite* if $\chi(G) \leqslant 2$. We denote such a graph by $G = (X, Y, E)$, where X and Y are independent sets in G with $X \cap Y = \emptyset$ and $X \cup Y = V$. The *bipartite complement* of a bipartite graph $G = (X, Y, E)$ is the graph $G^b = (X, Y, \{\{x, y\} : \{x, y\} \notin E, x \in X \text{ and } y \in Y\})$.

For all graphs G we have the inequalities $\alpha(G) \leqslant \kappa(G)$ and $\omega(G) \leqslant \chi(G)$. Moreover, we obtain from the definitions $\alpha(G) = \omega(G^c)$ and $\chi(G) = \kappa(G^c)$. The following definitions are due to Berge, see 'Introduction' in [4] for historical remarks.

Definition 2. $G = (V, E)$ is *α-perfect* if for all $W \subseteq V$ holds $\alpha(G[W]) = \kappa(G[W])$. $G = (V, E)$ is *ω-perfect* if for all $W \subseteq V$ holds $\omega(G[W]) = \chi(G[W])$.

Berge posed two conjectures according to this definition, the weaker one was proved by Lovász.

Theorem 1 (Perfect graph theorem [22]). *A graph G is α-perfect if and only if G is ω-perfect.*

Therefore α-perfect graphs are called *perfect* graphs for short. The stronger conjecture of Berge is still open.

Strong perfect graph conjecture (SPGC). A graph G is perfect if neither G nor G^c contain an induced subgraph isomorphic to a chordless cycle of odd length greater than four.

Now we are going to develop a similar theory for edge sets instead of vertex sets. There are other such kinds of perfectness, so for example, a generalised perfection by Chang and Nemhauser [5], K_i-perfectness due to Conforti et al. [8], neigbourhood perfectness defined by Lehel and Tuza [20], star perfectness considered by Nicolai [24], and line perfectness due to Trotter [29].

We define a pair of distinct edges e, f of a graph $G = (V,E)$ to be *dependent* in G if either e and f are incident with a common vertex in V, i.e. $e \cap f \neq \emptyset$, or the vertices of $e \cup f$ form a cycle of length four in G, possibly with chords.

Definition 3. $B \subseteq E$ is an *edge-clique* in $G = (V,E)$ if every pair of distinct edges of B is dependent in G. $M \subseteq E$ is an *independent set of edges* in $G = (V,E)$ if no pair of distinct edges of M is dependent in G.

$$\alpha_e(G) = \max\{|M| : M \text{ is an independent set of edges in } G\},$$

$$\omega_e(G) = \max\{|B| : B \text{ is an edge-clique in } G\},$$

$$\chi_e(G) = \min\{k : E \text{ is covered by independent sets of edges } M_1, M_2, \ldots, M_k \text{ in } G\},$$

$$\kappa_e(G) = \min\{k : E \text{ is covered by edge-cliques } B_1, B_2, \ldots, B_k \text{ in } G\}.$$

These definitions are motivated by their meaning in case of bipartite graphs G. The edges of a complete bipartite subgraph of G form an edge-clique of G, also called a *biclique* in bipartite framework. An independent set of edges in a graph G forms a matching in G without alternating cycles of length four, i.e. each cycle of length four in G contains at most one edge of an independent set of edges in G. For short we denote an independent set of edges aC_4fm (alternating C_4-free matching) in bipartite framework.

As in case of the parameters α, χ, κ and ω we have for all graphs G the inequalities $\alpha_e(G) \leqslant \kappa_e(G)$ and $\omega_e(G) \leqslant \chi_e(G)$. In contrast there is no analogue to $\alpha(G) = \omega(G^c)$ and $\chi(G) = \kappa(G^c)$. We obtain for the bipartite graph P_6 with bipartite complement $P_4 + P_2$ the parameters $\alpha_e(P_6) = 3$, $\omega_e(P_6^b) = 2$, $\omega_e(P_6^c) = 6$ and $\chi_e(P_6) = 2$, $\kappa_e(P_6^b) = 3$, $\kappa_e(P_6^c) = 2$. The reason is that we consider edges here. An edge disappears in the (bipartite) complement, a vertex remains present.

Definition 4. $G = (V,E)$ is α_e*-perfect* if for all $W \subseteq V$ holds $\alpha_e(G[W]) = \kappa_e(G[W])$. $G = (V,E)$ is ω_e*-perfect* if for all $W \subseteq V$ holds $\omega_e(G[W]) = \chi_e(G[W])$.

We have no analogy to the Perfect Graph Theorem in case of edge-perfectness. For the cube graph Q_3 (see Fig. 1) the values $\alpha_e(Q_3) = 3$ and $\omega_e(Q_3) = \chi_e(Q_3) = \kappa_e(Q_3) = 4$.

The bipartite graphs Q_3 and $Q_3 - e$ (see Fig. 2) are minimal α_e-imperfect, but ω_e-perfect. There are other bipartite graphs with this property. However, the known ω_e-imperfect graphs are α_e-imperfect too. So we pose the following conjecture.

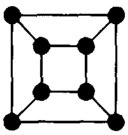

Fig. 1. The graph Q_3.

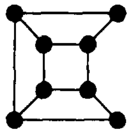

Fig. 2. The graph $Q_3 - e$.

Conjecture 1. Bipartite α_e-perfect graphs are ω_e-perfect.

For the nonbipartite graph R depicted in Fig. 3 we compute $\alpha_e(R) = 3, \kappa_e(R) = 4$, $\omega_e(R) = 5$ and $\chi_e(R) = 6$. This graph is both minimal α_e-imperfect and minimal ω_e-imperfect.

In what follows, we deal with a stronger version of edge-perfectness close to perfectness. This enables us to adapt methods from the theory of perfect graphs and minimal imperfect graphs.

Consider a graph $G = (V,E)$ and a set $F \subseteq E$. We define the dependence graph $D(G)$ and the parameters $\alpha_e, \omega_e, \chi_e$ and κ_e with respect to F.

Definition 5.

$$D(G) = (E, \{\{e,f\}\colon e,f \in E \text{ and } e \text{ and } f \text{ are dependent in } G\},$$
$$\alpha_e(G,F) = \max\{|M \cap F|\colon M \text{ is an independent set of edges in } G\},$$
$$\omega_e(G,F) = \max\{|B \cap F|\colon B \text{ is an edge-clique in } G\},$$
$$\chi_e(G,F) = \min\{k\colon F \text{ is covered by independent sets of edges } M_1, M_2, \ldots, M_k \text{ in } G\},$$
$$\kappa_e(G,F) = \min\{k\colon F \text{ is covered by edge-cliques } B_1, B_2, \ldots, B_k \text{ in } G\}.$$

In what follows, $D(G)$ is called the *dependence graph* of G.

Lemma 1. *Let $G = (V,E)$ be a graph. The following conditions are equivalent:*
1. *For all $F \subseteq E$ holds $\alpha_e(G,F) = \kappa_e(G,F)$;*
2. *For all $F \subseteq E$ holds $\omega_e(G,F) = \chi_e(G,F)$;*
3. *$D(G)$ is perfect.*

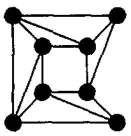

Fig. 3. The graph R.

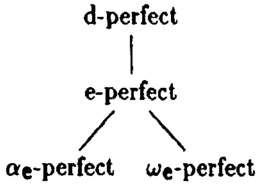

Diagram 1. Differnt types of edge-perfectness. The upper type implies the lower one.

Proof. It is easy to verify the truth of the following statements for all graphs $G = (V,E)$ and $F \subseteq E$:

$$\alpha_e(G,F) = \alpha(D(G)[F]), \qquad \kappa_e(G,F) = \kappa(D(G)[F]),$$
$$\omega_e(G,F) = \omega(D(G)[F]), \qquad \chi_e(G,F) = \chi(D(G)[F]).$$

Now one can easily prove the lemma using the Perfect Graph Theorem. □

Definition 6. A graph $G = (V,E)$ is d-*perfect* if $D(G)$ is perfect.

Lemma 2. *Let $G = (V,E)$ be a d-perfect graph. Then G is α_e-perfect and ω_e-perfect.*

Proof. Again it is easy to verify the truth of the following statements for all graphs $G = (V,E)$, $W \subseteq V$ and $F = E(G[W])$:

$$\alpha_e(G,F) = \alpha(D(G)[F]), \qquad \kappa_e(G,F) = \kappa(D(G)[F]),$$
$$\omega_e(G,F) = \omega(D(G)[F]), \qquad \chi_e(G,F) = \chi(D(G)[F]). \qquad \square$$

Consider the graph Q_3: this graph is ω_e-perfect, but $D(Q_3)$ contains a C_5. Hence, ω_e-perfectness does not imply d-perfectness. In what follows, a graph is said to be *e-perfect* if it is both α_e- and ω_e-perfect. Diagram 1 shows the relation between different types of edge-perfectness.

2. Perfectness preserving operations

2.1. One vertex extensions

Definition 7. Let $G = (V,E)$ be a graph and let $x \in V$ and $y \notin V$ be vertices. We define operations PV (*pendant vertex*), FT (*false twin*) and TT (*true twin*) by

$V' = V \cup \{y\}$ and

$$(V', E_{\mathrm{pv}}) = \mathrm{PV}(G,x,y) \iff E_{\mathrm{pv}} = E \cup \{\{x,y\}\},$$
$$(V', E_{\mathrm{ft}}) = \mathrm{FT}(G,x,y) \iff E_{\mathrm{ft}} = E \cup \{\{y,z\} : \{x,z\} \in E\},$$
$$(V', E_{\mathrm{tt}}) = \mathrm{TT}(G,x,y) \iff E_{\mathrm{tt}} = E_{\mathrm{ft}} \cup \{\{x,y\}\}.$$

Lemma 3. *The classes of α_{e}-perfect and ω_{e}-perfect graphs are closed under the pendant vertex operation.*

Proof. Let $G = (V,E)$ be an arbitrary subgraph of an α_{e}-perfect graph, let $x \in V$ and $y \notin V$ be vertices and $G' = \mathrm{PV}(G,x,y)$. Let us first assume that $\alpha_{\mathrm{e}}(G) = \alpha_{\mathrm{e}}(G-x)$. Then $\{x,y\}$ is independent from all edges of $G-x$ and hence $\alpha_{\mathrm{e}}(G') \geqslant \alpha_{\mathrm{e}}(G-x)+1$. And $\{\{x,y\}\} \cup I_G(x)$ is an edge-clique of G' and hence $\kappa_{\mathrm{e}}(G') \leqslant \kappa_{\mathrm{e}}(G-x)+1$. On the other hand, $G-x$ is α_{e}-perfect and this implies $\alpha_{\mathrm{e}}(G-x) = \kappa_{\mathrm{e}}(G-x)$. Using $\kappa_{\mathrm{e}}(G') \geqslant \alpha_{\mathrm{e}}(G')$ we conclude $\alpha_{\mathrm{e}}(G') = \kappa_{\mathrm{e}}(G') = \alpha_{\mathrm{e}}(G)+1$. Next we assume $\alpha_{\mathrm{e}}(G) \neq \alpha_{\mathrm{e}}(G-x)$. Then we have $\alpha_{\mathrm{e}}(G) = \alpha_{\mathrm{e}}(G-x)+1$ and we obtain similar to the first case $\alpha_{\mathrm{e}}(G') = \kappa_{\mathrm{e}}(G') = \alpha_{\mathrm{e}}(G)$. Consequently, α_{e}-perfect graphs are closed under adding pendant vertices.

Now let $G = (V,E)$ be an arbitrary subgraph of an ω_{e}-perfect graph, x, y and G' as before. Two different edges $\{x,y\}$ and $\{x',z\}$ of G' are dependent if and only if adjacent, i.e. $x = x'$ and $z \in N_G(x)$. Hence $\omega_{\mathrm{e}}(G') = \max(\omega_{\mathrm{e}}(G), d_G(x)+1)$ and $\chi_{\mathrm{e}}(G') = \max(\chi_{\mathrm{e}}(G), d_G(x)+1\}$. This completes the proof of the lemma. □

Clearly the hull with respect to PV of the K_1 is the class of trees.

Corollary 1. *Trees are e-perfect.*

Lemma 4. *The class of α_{e}-perfect graphs is closed under false twin and true twin operations.*

Proof. Let $G = (V,E)$ be an arbitrary subgraph of an α_{e}-perfect graph, let $x \in V$ and $y \notin V$ be vertices and $G' = \mathrm{FT}(G,x,y)$. For each edge-clique cover $\mathscr{F}$ of G the set $\mathscr{F}' = \{F' : F \in \mathscr{F}\}$ with $F' = F \cup \{\{y,z\} : \{x,z\} \in F\}$ is an edge-clique cover of G', hence $\kappa_{\mathrm{e}}(G') \leqslant \kappa_{\mathrm{e}}(G)$. On the other hand, we have by the α_{e}-perfectness of G that $\kappa_{\mathrm{e}}(G) \leqslant \alpha_{\mathrm{e}}(G) \leqslant \alpha_{\mathrm{e}}(G') \leqslant \kappa_{\mathrm{e}}(G')$ and hence $\alpha_{\mathrm{e}}(G') = \kappa_{\mathrm{e}}(G') = \alpha_{\mathrm{e}}(G)$. Consequently, α_{e}-perfect graphs are closed under taking false twins.

Now let G, x and y as before but $G' = \mathrm{TT}(G,x,y)$. We proceed as in the proof of Lemma 3. If $\alpha_{\mathrm{e}}(G) = \alpha_{\mathrm{e}}(G-x)$ then $\{x,y\}$ is independent from all edges of $G-x$ and $\{\{x,y\}\} \cup \{\{x,z\},\{y,z\} : z \in N_G(x)\}$ is an edge-clique of G'. We conclude $\alpha_{\mathrm{e}}(G') = \kappa_{\mathrm{e}}(G') = \alpha_{\mathrm{e}}(G)+1$. For $\alpha_{\mathrm{e}}(G) = \alpha_{\mathrm{e}}(G-x)+1$ we compute $\alpha_{\mathrm{e}}(G') = \kappa_{\mathrm{e}}(G') = \alpha_{\mathrm{e}}(G)$. This completes the proof of the lemma. □

The hull with respect to PV, FT and TT of the K_1 is the class of distance hereditary graphs, see [2].

Corollary 2. *Distance hereditary graphs are* α_e*-perfect.*

2.2. Bonding

Definition 8. A graph G is a S-bonding of two graphs G_1 and G_2 if and only if $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$, $S = V(G_1) \cap V(G_2)$ and $G_1[S] = G_2[S]$.

It is well-known that perfect graphs are closed under S-bonding if S is a clique ('clique bonding'). Or, equivalently, no minimal imperfect graph has a clique cutset, see for example [25]. A more general result is due to Chvatál [7].

Theorem 2 (Star-cutset lemma). *No minimal imperfect graph has a star-cutset.*

A similar statement was proved by Tucker [30].

Theorem 3. *No minimal imperfect graph has an independent cutset, unless it is an odd hole.*

Hereby an *odd hole* is an induced chordless cycle of odd length. Again, the assertion of the theorem corresponds with the existence of a perfectness preserving operation, namely, perfect graphs are closed under S-bonding if S is an independent set, unless an odd hole is created ('independent set bonding'), see [9] also.

In this subsection a translation to d-perfect graphs is given. Note that for $G = (V,E)$ with $F \subseteq E$ the graph $D(G)[F]$ is minimal imperfect if and only if $\alpha_e(G,F) < \kappa_e(G,F)$ and $\omega_e(G,F) < \chi_e(G,F)$, but for all $F' \subset F$ holds $\alpha_e(G,F') = \kappa_e(G,F')$ and $\omega_e(G,F') = \chi_e(G,F')$.

Lemma 5. *Let $G = (V,E)$ be a graph, $F \subseteq E$ such that $D(G)[F]$ is minimal imperfect but is not an odd hole. Then a set $S \subseteq F$ with the following properties does not exist:*

- *S is an independent set of edges;*
- *there exist at least two edges in different connected components of $(V, F \setminus S)$.*

Proof. To establish the lemma, we prove that if (G,F) admits a set S with the two properties above, then $D(G)[F]$ has an independent cutset. Since $D(G)[F]$ is a minimal imperfect graph which is not an odd hole, that is not possible.

Let V_1 and V_2 be the vertex sets of any two connected components of $(V, F \setminus S)$ containing at least one edge each. Then V_1 and V_2 determine in $D(G)[F \setminus S]$ two subgraphs denoted, for simplicity, by $D(V_1)$ and $D(V_2)$. On the other hand, the edges

Table 1
Parameters for cycles.

G	$\lvert E(G)\rvert$	$\omega_e(G)$	$\chi_e(G)$	$\alpha_e(G)$	$\kappa_e(G)$
C_3	3	3	3	1	1
C_4	4	4	4	1	1
C_4^c	2	1	1	2	2
C_5	5	2	3	2	3
C_6	6	2	2	3	3
C_6^c	9	4	5	2	3
C_n	n	2	$2 + p(n)$	$\lfloor n/2 \rfloor$	$\lceil n/2 \rceil$
C_n^c	$n(n-3)/2$	$n(n-5)/2$	$n(n-5)/2$	2	$3 + p(n)$

in S form an independent set of $D(G)$. Obviously, $D(V_1)$ and $D(V_2)$ are connected graphs (since for $i = 1,2$ the graphs $(V, F \setminus S)[V_i]$ are connected and two adjacent edges are dependent). Moreover, the subgraph of $D(G)$ induced by the union of their vertex sets is not connected.

Suppose the contrary. Then there exist two vertices e of $D(V_1)$ and f of $D(V_2)$ such that $\{e, f\}$ is an edge of $D(G)$, i.e. the edges e and f are dependent in G. If they have a common vertex, then in G there is a vertex in V_1 and a vertex in V_2 joined by an edge not in S, a contradiction. If the two edges e and f are opposite edges in a cycle of length four, then we obtain the same conclusion, unless the other two edges of this cycle are both in S. But in this case S is not an independent set of edges, a contradiction.

We deduce that no adjacent vertices e of $D(V_1)$ and f of $D(V_2)$ exist, therefore S is an independent cutset of $D(G)$. By theorem 3, using the facts that $D(G)$ is minimal imperfect and is not an odd hole, we have a contradiction. □

Clearly, we can deduce from the lemma a d-perfectness preserving operation 'independent edge set bonding'. The class of d-perfect graphs is not closed under 'edge clique bonding . An example is the C_6^c, another one is given in Fig. 4.

3. Edge perfect graphs and perfect graphs

First in this section we give in Table 1 some values of parameters for cycles. Next we show how to compute these values for cycles of length greater than six (suppose $n > 6$ and define the parity of n by $p(n) = 1$ if $n \equiv 1 \pmod 2$ and $p(n) = 0$ if $n \equiv 0 \pmod 2$).

We consider the graph $C_n^c = (\{1, 2, \ldots n\}, \{\{i,j\} : 1 \leqslant i < j \leqslant n \text{ and } |i-j| \not\equiv 1 \pmod n\})$ for $n > 6$. We denote the set of *short edges* of G by $S = \{\{i,j\} : 1 \leqslant i < j \leqslant n \text{ and } |i-j| \equiv 2 \pmod n\})$. Two edges of C_n^c not in S are dependent since they share a common vertex or they are included in a cycle of length four. Two edges

e and f are independent if at least one is a short edge, say $e = \{i-1, i+1\} \in S$ (for a suitable i) and the other one crosses the first one, i.e. $f = \{i,j\}$ for $1 \leqslant j \leqslant i-2$ or $i+2 \leqslant j \leqslant n$. We deduce $\alpha_e(C_n^c) = 2$ and $\chi_e(G_n^c) = n(n-5)/2$ since C_n^c admits only n disjoint maximal independent sets, each of size two.

We consider the set of nonshort edges $B = \{\{i,j\} : 1 \leqslant i,j \leqslant n$ and $|i-j| \not\equiv 2 \pmod{n}\}$. B is an edge-clique of C_n^c and for $n > 6$, B is the maximum edge-clique. To prove this we consider an egde clique C of C_n^c containing an edge e of S, w.l.o.g. $e = \{1,3\}$. Let $D = \{\{2,j\}: 4 \leqslant j \leqslant n\}$ the set of edges crossing e. We have $D \cap C = \emptyset$. Now we distinguish between two cases. First assume e is the only short edge in C. We conclude $|C| \leqslant |E(C_n^c)| - |D| - |S \setminus \{e\}| = \frac{1}{2}(n^2 - 7n + 8)$. Now assume C contains two short edges e and f with sets D and F of crossing edges, respectively. Then we conclude $|C| \leqslant |E(C_n^c)| - |D \cup F| = \frac{1}{2}(n^2 - 7n + 9)$. In both cases we have $|C| \leqslant \frac{1}{2}(n^2 - 7n + 9)$ in contrast to $|B| = \frac{1}{2}(n^2 - 5n)$. So we have $\omega_e(C_n^c) = n(n-5)/2$ for $n > 5$.

Now we consider coverings of $E(C_n^c)$ by edge-cliques. It is easy to see that we need for even n at least two edge-cliques to cover all the short edges and an additional one to cover the remaining edges. For odd n we need at least three edge-cliques to cover the short edges and an additional one for the remaining edges. In order to show that this is also sufficient, we split S into two cycles of pairwise dependent edges if n is even and add B to this partial covering. If n is odd we cover S by three edge-cliques. So we have for n even $\kappa_e(C_n^c) = 3$ and for n odd $\kappa_e(C_n^c) = 4$. We give an other partition of $E(C_n^c)$ into three edge-cliques in case of even n:

$$B_1 = \{\{i,j\}: i \equiv j \equiv 1 \pmod{2}\},$$
$$B_2 = \{\{i,j\}: i \equiv j \equiv 0 \pmod{2}\},$$
$$B_3 = \{\{i,j\}: i + j \equiv 1 \pmod{2}\}.$$

Corollary 3. *e-perfect graphs are Berge graphs.*

Conjecture 2. *e-perfect graphs are perfect graphs.*

4. Classes of d-perfect graphs

First we mention here some special classes of perfect graphs. For a general survey see [3] and [13]. For the graph classes listed below several characterizations are known. Moreover, special polynomial-time algorithms exist to compute the parameters α, χ, κ and ω of graphs taken from these classes. Here we restrict ourselfs to some references.

The first class are the *chordal graphs*. See [13] for characterizations, algorithms and further references. The class of chordal graphs has two important subclasses, the *strongly chordal graphs*, see [11], and *split graphs*, see [13].

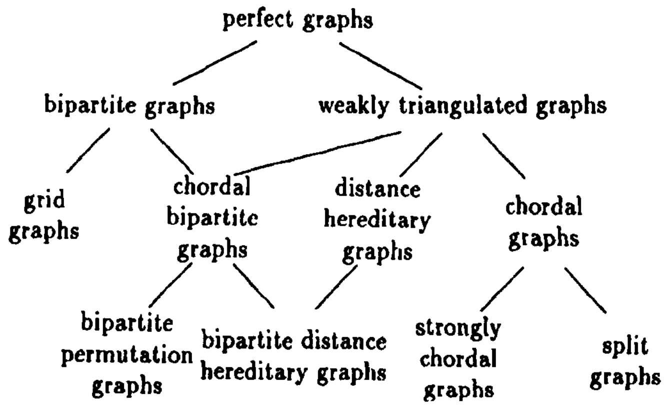

Diagram 2. Containment of graph classes. The upper class contains the lower one.

Secondly, we mention Hayward's class of *weakly triangulated graphs*, a generalisation of chordal graphs, see [16,17]. The graphs both weakly triangulated and bipartite form the class of *chordal bipartite graphs*. This class is not included in the class of chordal graphs.

Sometime we need *distance hereditary graphs* and *bipartite distance hereditary graphs*. Distance hereditary graphs are weakly triangulated graphs. The graphs both bipartite and distance hereditary form the class of *bipartite distance hereditary graphs*. See [19,2,1]. Some details on chordal bipartite graphs an bipartite distance hereditary graphs are given in Section 8.1.

Last we refer to *bipartite permutation graphs*, a subclass of the chordal bipartite graphs, see [27].

Diagram 2 shows the containment of the mentioned clases.

Now we continue with some lemmas showing that distance hereditary graphs and bipartite permutation graphs are d-perfect graphs.

Lemma 6 ([23]). *Let G be a bipartite distance hereditary graph. Then $D(G)$ is a chordal graph.*

Lemma 7 ([23]). *Let G be a distance hereditary graph. Then $D(G)$ is a weakly triangulated graph.*

Lemma 8. *Let $G = (X, Y, E)$ be a bipartite permutation graph. Then $D(G)$ is a weakly triangulated graph.*

Proof. Consider a bipartite permutation graph $G = (X, Y, E)$ with a strong ordering of the vertices $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_m\}$. Moreover, consider a set $F \subseteq E$ of edges and denote the graph, induced in the dependence graph of G, by $D(G)[F]$. It suffices to prove that $D(G)[F]$ is a clique or contains a two-pair. We distinguish between four cases.

Case 1: *F contains less than two edges.* $D(G)[F]$ is a clique.

Case 2: *F contains at least two vertices in different connected components of* $D(G)[F]$. We obtain a two-pair of $D(G)[F]$ by taking two edges of F lying in two distinct components of $D(G)[F]$. In the following cases we assume that G is connected and each vertex of G is incident with at least one edge of F since we are looking for a two pair in $D(G)[F]$.

Case 3: *For all* i, j *with* $2 \leqslant i \leqslant n$, $2 \leqslant j \leqslant m$, $\{x_{i-1}, y_{j-1}\} \in F$ *and* $\{x_i, y_j\} \in F$ *the vertices* $x_{i-1}, x_i, y_{j-1}, y_j$ *induce a complete cycle of length four in* G. In this case $D(G)[F]$ is a clique since G is a complete bipartite graph. We show the completeness by induction on n and m. For $n = m = 1$ this follows from the connectedness of G. Now we suppose $G - x_1$ is a complete bipartite subgraph of G. Since G is connected x_1 as a neighbour, say y_l. By the definition of strong ordering x_1 is adjacent to $y_1, \ldots, y_{l-1}$ and by the assumption of this case x_1 is adjacent to $y_{l+1}, \ldots, y_m$.

Case 4: *There exist* i *and* j *with* $2 \leqslant i \leqslant n$, $2 \leqslant j \leqslant m$, $\{x_{i-1}, y_{j-1}\} \in F$ *and* $\{x_i, y_j\} \in F$ *and the vertices* $x_{i-1}, x_i, y_{j-1}, y_j$ *do not induce a cycle of length four in* G. We show $\{x_{i-1}, y_{j-1}\}$ and $\{x_i, y_j\}$ are a two-pair of $D(G)[F]$. We assume in contradiction there is a path P in $D(G)[F]$ of length greater than two connecting $\{x_{i-1}, y_{j-1}\}$ and $\{x_i, y_j\}$. Therefore, we distinguish two subcases.

Case 4.1: *P contains a vertex* $\{x_k, y_l\}$ *with* $(k-i)(l-j) < 0$. In this case P has two chords $\{\{x_k, y_l\}, \{x_i, y_j\}\}$ and $\{\{x_k, y_l\}, \{x_{i-1}, y_{j-1}\}\}$ by the property of the strong ordering.

Case 4.2: *P does not contain any vertex* $\{x_k, y_l\}$ *with* $(k-i)(l-j) < 0$. Now P contains two adjacent vertices $\{x_k, y_l\}$ and $\{x_p, y_q\}$ with $k < i-1$, $l < j-1$, $p > i$ and $q > j$. By the adjacency and the property of the strong ordering the vertices $\{x_i, y_j\}$, $\{x_{i-1}, y_{j-1}\}$, $\{x_k, y_l\}$ and $\{x_p, y_q\}$ form a $K_4 - e$ in $D(G)$ and P has chords. This completes the proof. □

Lemma 9. *A bipartite graph* $G = (X, Y, E)$ *without induced subgraph isomorphic to* C_4 *is d-perfect.*

Proof. In a straightforward matter we compute the following values for any $F \subseteq E$, see Section 6.5 also:

- $\omega_e(G,F) = \max\{|\{w\colon \{v,w\} \in F\}| : v \in X \cup Y\}$, the maximum degree of a vertex in $X \cup Y$ in (X, Y, F),
- $\chi_e(G,F) = \omega_e(G,F)$,
- $\kappa_e(G,F) = \min\{|W| : W \subseteq X \cup Y \text{ and } \bigwedge e \in F \quad [e \cap W \neq \emptyset]\}$, the minimum cardinality of a vertex cover in (X, Y, F), and
- $\alpha_e(G,F) = \max\{|M| : M \subset F \text{ and } \bigwedge e,f \in M \quad [e = f \vee e \cap f = \emptyset]\}$ the maximum cardinality of an ordinary matching in (X, Y, F).

By Kőnig's theorem we obtain $\kappa_e(G,F) = \alpha_e(G,F)$. Alternatively, $D(G)$ is the usual line graph of G, since G does not contain cycles of length four. We are done since line graphs of bipartite graphs are perfect. □

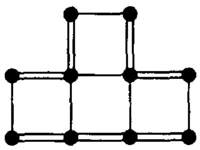

Fig. 4. A chordal bipartite grid graph. The double lines indicate a C_7 in the dependence graph.

Definition 9. The *complete n,m-grid* is the graph $G_{n,m} = (V_{n,m}, E_{n,m})$ with $V_{n,m} = \{(i,j)\colon 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant m\}$ and $E_{n,m} = \binom{V_{n,m}}{2} \cap \{\{(i,j),(k,l)\}\colon |i-k| + |j-l| = 1\}$. A graph $G = (V,E)$ is a *grid graph* if there are positive integers n and m such that $V \subseteq V_{n,m}$ and $E \subseteq E_{n,m} \cap \binom{V}{2}$.

Lemma 10. *Each grid graph G is both a bipartite graph and a ω_e-perfect graph.*

Proof. It is easy to see that G is a bipartite graph. If G contains no C_4 the lemma follows from the preceding one. So we may assume that G contains a C_4.

First, we consider a partition of $E_{n,m}$ in four aC4fms:

$$M_1 = \{\{(i,j),(i,j+1)\}\colon i+j \equiv 0 \pmod 2\},$$
$$M_3 = \{\{(i,j),(i+1,j)\}\colon i+j \equiv 0 \pmod 2\},$$
$$M_2 = \{\{(i,j),(i,j+1)\}\colon i+j \equiv 1 \pmod 2\},$$
$$M_4 = \{\{(i,j),(i+1,j)\}\colon i+j \equiv 1 \pmod 2\}.$$

Now consider $G = (X,Y,E)$. By the above partition of $E_{n,m}$ we have $\chi_e(G) \leqslant 4$. On the other hand, G contains a C_4, and this implies $\omega_e(G,F) = 4$. Last consider $G[W]$ for a set $W \subset X \cup Y$. Then $(X \cap W, Y \cap W, E \cap \binom{W}{2})$ is a grid graph too and we are done by induction. □

Remark 1. Grid graphs and chordal bipartite graphs are in general not d-perfect, see the example in Fig. 4.

5. Inflations of 3-regular graphs

A graph is 3-regular if each vertex is of degree three. We get the inflation G_t of a 3-regular graph $G = (V,E)$ by replacing each vertex by a triangle, formally $G_t = (V_t, E_t)$ with $V_t = \{(v,e)\colon e \in E \text{ and } v \in V \cap e\}$ and $E_t = \{\{(v,e),(w,f)\}\colon e = f = \{v,w\}$ or $\{v\} = \{w\} = e \cap f\}$. An edge $\{(v,\{v,w\}),(w,\{v,w\})\}$ of G_t is called an *original edge*. All other edges of G_t are called *triangle edges*.

Lemma 11. *Suppose $G = (V,E)$ is a 3-regular graph. Then G_t is 3-regular, perfect and does not contain a cycle of length four.*

Proof. It is easy to see that G_t is 3-regular. Consider a chordless cycle C_t of length at least four in G_t. Then C_t uses alternately edges of $\{\{(v,e),(v,f)\} : v = e \cap f\}$ (triangle edges) and $\{\{(v,e),(w,e)\} : e = \{v,w\}\}$ (original edges). Hence C_t is of even length and corresponds to a cycle C in G of half its length. So G_t does not contain a C_4 or C_{2k+1} for $k \geqslant 2$. But a C_{2k+1}^c for $k \geqslant 3$ contains a C_4 and C_5^c is isomorphic to C_5. All the C_3's in G_t use triangle edges only. Hence G_t does not contain a cycle of length four with at least one chord, since each vertex of G_t is incident with exactly two triangle edges. In particular, G_t does not contain a K_4. Thus, our class of graphs is K_4-free. Tucker [31] has shown that the SPGC is valid for K_4-free graphs. Hence, our class of graphs is perfect. □

Lemma 12. *Suppose $G = (V,E)$ is a 3-regular connected graph with n vertices. Then we have*

- $\omega_e(G_t) = 3$,
- $\alpha_e(G_t) = 3n/2$,
- $\chi_e(G_t) = \chi'(G)$, *where $\chi'(G)$ denotes the chromatic index of G, and*
- $\kappa_e(G_t) = \lceil 7n/4 \rceil$.

Proof. We have $\omega_e(G_t) = 3$ since G_t is 3-regular without cycle of length four by the previous lemma.

The original edges form an independent set $\{\{(v,e),(w,e)\} : e = \{v,w\}\}$ of cardinality $m = 3n/2$ in G_t. Consider an arbitrary independent set $M \subseteq E_t$ in G_t. M contains at most one edge from each triangle in G_t. If M contains k triangle edges then M contains at most $m - k$ original edges since each triangle edge is incident with two original edges. But the original edges incident with a triangle edge in M cannot belong to M since M is a matching. Hence $|M| \leqslant m$.

Consider a matching M of G and define M_t to be the set of original edges of G_t corresponding to edges in M together with the triangle edges not incident with these edges, $M_t = \{\{(v,e),(w,e)\} : \{v,w\} = e \in M\} \cup \{\{(v,g),(v,f)\} : \{v\} = g \cap f$ and $g, f \notin M$, but there is an edge $e \ni v$ with $e \in M\}$. M_t is an independent set of edges in G_t. Each triangle edge $\{(v,g),(v,f)\}$ shares an independent set M_t with the original edge $\{(v,e),(w,e)\}$, $e = \{v,w\} \notin \{g,f\}$, incident with the third vertex of the triangle. This implies $\chi_e(G_t) \leqslant \chi'(G)$.

On the other hand, consider a matching M_t belonging to a minimum cover by independent sets of edges in G_t and define $M = \{\{v,w\} : \{(v,\{v,w\}),(w,\{v,w\})\} \in M_t\}$. Suppose M is not a matching of G. Then M_t contains two original edges in M_t incident with a common triangle edge. Then the edges of this triangle belong to three different sets of the considered cover. So we have $\chi_e(G_t) \geqslant 4$ and we are done by Vizing's theorem. If M is a matching of G we have $\chi_e(G_t) \geqslant \chi'(G)$.

Now we prove $\kappa_e(G_t) = \lceil 7n/4 \rceil$. The maximal edge-cliques usable to cover the edges of G_t are either three triangle edges forming a triangle or one original edge and two incident triangle edges forming a 3-star ($K_{1,3}$). To cover all the original edges we need $m = 3n/2$ 3-stars. The remaining triangle edges can be covered by triangles.

A triangle is covered by 3-stars if at least two of the incident original edges are covered by 3-stars covering edges of the considered triangle too. We can represent this by an orientation of $G = (V,E)$. An edge $\{v,w\} \in E$ is directed from v to w if the 3-star covering $\{v,w\}$ covers two edges form the triangle corresponding to w. So the problem of finding a minimum edge-clique cover of G_t is equivalent to construct an orientation of $G = (V,E)$ with a minimum number of vertices with indegree less than two. For a fixed orientation we denote by n_i the number of vertices in V with indegree i, $0 \leqslant i \leqslant 3$.

Claim 1. *There is an orientation of G with $n_1 + n_3 \leqslant 1$.*

This claim implies $\kappa_e(G_t) = \lceil 7n/4 \rceil$ since the following equations hold:

$$\kappa_e(G_t) = n_0 + n_1 + m,$$
$$n = n_0 + n_1 + n_2 + n_3,$$
$$m = n_1 + 2n_2 + 3n_3,$$
$$2m = 3n,$$
$$0 \leqslant n_0, n_1, n_2, n_3.$$

For fixed n (and m) we consider these conditions as a linear programming problem without background. Then the minimum integer solution is $\kappa_e(G_t) = \lceil 7n/4 \rceil$. This value can be reached for $n_1 + n_3 \leqslant 1$.

Proof of claim 1. We choose an arbitrary orientation of G. We say a vertex $v \in V$ is of type 1 if it has indegree one or three and v is of type 2 otherwise. Suppose there are two vertices v and w of type 1. We choose a path joining v and w and change the orientation along this path. Thereby v and w become of type 2 while all the other vertices keep their type. We are done by induction. This way we are able to construct an optimal orientation in polynomial time. □

Lemma 13. *Inflations of 3-regular graphs can be recognized in polynomial time. For the inflation G_t of a 3-regular graph $\omega_e(G_t)$, $\alpha_e(G_t)$ and $\kappa_e(G_t)$ can be computed in polynomial time, but to decide $\{(G,k)\colon \chi_e(G_t) \leqslant k\}$ remains NP-complete.*

Proof. The proof is straightforward using the previous lemma and the NP-completeness result concerning χ' for 3-regular graphs of Holyer in [18]. □

6. Complexity results

Here we summarise some corollaries from the previous lemmas and from the other papers. We consider the complexity of computing the parameters α_e, ω_e, χ_e and κ_e for special classes of graphs.

Table 2
Complexity overview.

	MM	MEC	MC	ECC
Graphs in general	N		N	N
Bipartite graphs	N		N	N
Chordal bipartite graphs	N	P		N
Bipartite distance hereditary graphs	P	P	P	P
Bipartite permutation graphs	P	P	P	P
C_4-free gaphs in general			N	N
Bipartite C_4-free gaphs	P	P	P	P

First we can compute these parameters in polynomial time for all classes of d-perfect graphs. Therefore, first compute on input G the dependence graph $D(G)$ and use an algorithm for perfect graphs to determine $\alpha(D(G))$ or $\chi(D(G))$ to get $\alpha_e(G) = \kappa_e(G)$ or $\chi_e(G) = \omega_e(G)$, respectively, see [14]. If $D(G)$ is weakly triangulated or chordal, simpler algorithms exist, see Section 4.

In general, it seems to be impossible to compute these parameters in polynomial time for an arbitrary graph. To use the notion of NP-completeness we define decision problems in the manner of Garey and Johnson in [12]:

$$\mathrm{MM} = \{(G,k)\colon \alpha_e(G) \leqslant k\}, \qquad \mathrm{MEC} = \{(G,k)\colon \omega_e(G) \leqslant k\},$$
$$\mathrm{MC} = \{(G,k)\colon \chi_e(G) \leqslant k\}, \qquad \mathrm{ECC} = \{(G,k)\colon \kappa_e(G) \leqslant k\}.$$

We give an overview on the complexities in Table 2 and list the references in subsections. In this table 'N' means NP-completeness and 'P' solvability in polynomial time.

6.1. *Maximum independent set of edges (MM)*

The complexity of computing MM coincides with the complexity of the jump number problem for a transitive orientation of G if G is a chordal bipartite graph, see [6, 23]. Both problems are NP-complete for chordal bipartite graphs, see [23], and hence for bipartite graphs. These problems are solvable in polynomial time on convex bipartite graphs [10] and therefore for bipartite permutation graphs (see also [28]), and bipartite distance hereditary graphs.

6.2. *Maximum edge-clique (MEC)*

We are able to compute $\omega_e(G)$ in polynomial time for chordal bipartite graphs G. First we produce a perfect edge elimination scheme, cf. [13,15] and Lemma 15, of G.

Since a bisimplicial edge is contained in exactly one maximal edge-clique, we get a list of $m = |E(G)|$ edge-cliques too. We can extract the list of maximal edge-cliques from this list or we simply look for the maximum edge-clique in this list.

6.3. *Minimum cover of independent sets of edges (MC)*

The problem MC restricted to bipartite graphs turns out to be NP-complete. The proof is given Section 7.

6.4. *Minimum edge-clique cover (ECC)*

Moreover, the problem ECC restricted to chordal bipartite graphs is NP-complete. The proof is given in Section 8 and in [26]. For general C_4-free graphs this problem is NP-complete too. A simple proof was given implicitely in [12, pp. 54–55], original for VERTEX COVER, and the reduction graph used there is C_4-free. As mentioned above, this is a NP-completeness proof for ECC on general C_4-free graphs.

6.5. *Graphs without 4-cycles*

The considered problems simplify for bipartite C_4-free graphs, see Lemma 9 and its proof. In bipartite graphs each cycle of length four is an induced cycle. Hence, an ordinary matching in a bipartite C_4-free graph is an aC$_4$fm. The cardinality of a maximum edge-clique coincides with the maximum degree of a vertex in a bipartite C_4-free graph. An edge-clique cover corresponds to a vertex cover, see problem [GT1] in [12]. And the parameter χ_e coincides with the chromatic index for bipartite C_4-free graphs, see [OPEN 5] in [12].

This simple structure of an independent set of edges or an edge-clique generalises to graphs without induced subgraph isomorphic to C_4, $K_4 - e$ (the diamond) or K_4. Inflations of 3-regular graphs are such graphs, see Lemma 11. To determine χ_e remains NP-complete for these graphs, see Lemma 13. The problem VERTEX COVER becomes polynomial when restricted to bipartite graphs, see [32] (the complement of the set of vertices covering all edges is an independent set). For general C_4-free graphs VERTEX COVER is NP-complete (the reduction graph given in [12] Section 3.1.3 does not contain C_4, $K_4 - e$ or K_4).

7. NP-completeness of MC

In this section we consider the decision problems

$$\mathrm{MC} = \{(G, k)\colon \chi_e(G) \leqslant k\}$$

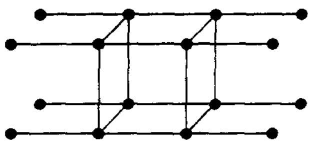

Fig. 5. Spiked cube.

and for fixed integers k

$$\mathrm{MC}_k = \{G\colon \chi_e(G) \leqslant k\}$$

restricted to bipartite graphs. MC_k is solvable in polynomial time for $k \leqslant 3$ since graphs G with $\chi_e(G) \leqslant 3$ have no induced subgraph isomorphic to C_4 and so χ_e coincides with the maximum degree.

In what follows, we prove the NP-completeness of MC_4 restricted to bipartite graphs.

Conjecture 3. *MC_k restricted to bipartite graphs is NP-complete for all $k \geqslant 4$.*

First we consider the graph in Fig. 5 called *spiked cube*.

The edges of the spiked cube incident with a vertex of degree one are called *spikes*. Two spikes are called *opposite* if their vertices of degree four do not share a common C_4.

Lemma 14. *Suppose $\mathcal{M}$ is a partition of the edge set of the spiked cube in alternating C_4-free matchings. Then $|\mathcal{M}| \geqslant 4$ and there is such a partition with $|\mathcal{M}| = 4$. Moreover, if $|\mathcal{M}| = 4$ the following holds:*

- *opposite spikes share a common matching $M \in \mathcal{M}$;*
- *each matching $M \in \mathcal{M}$ contains exactly two spikes.*

The proof by exhaustive search is left to the reader. Fig. 6 shows a typical partition of the edge set in four $\mathrm{aC_4fms}$.

Now we begin to describe a reduction of SATISFIABILITY, the well known NP-complete problem [LO1] of [12], to MC_4. We get the reduction graph by component design. First we define a *satisfaction test component* to be a spiked cube. Next we describe a *truth assignment component*. This component consists of six spiked cubes connected by identification of spikes as shown in Fig. 7.

The last component we need is called *colour base*. A colour base is created from at least three spiked cubes connected to a ring with respect to the bipartition (like in case of the truth assignment component). Hence a truth assignment component is a colour base on six spiked cubes.

Let $B = (W, A)$ be a colour base and $M \subset A$ an independent set of edges. In what follows, we choose for each pair of neighboured spikes cubes in the colour base one edge to be in M which was created by identifying spikes. Now we replace this edge

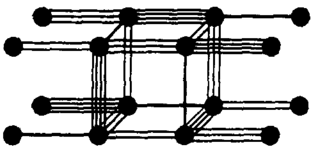

Fig. 6. Partition into four aC_4fms.

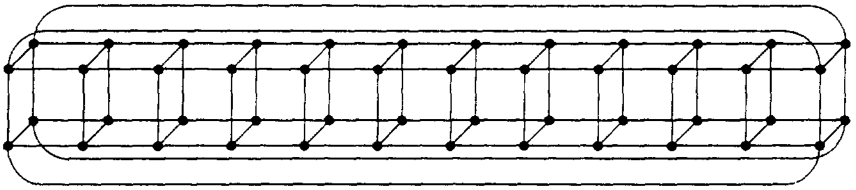

Fig. 7. Truth assignment component.

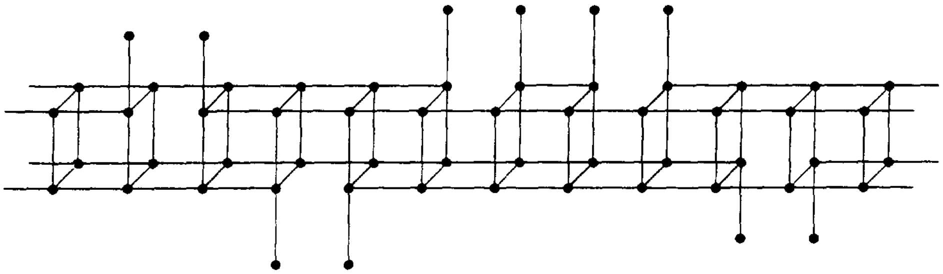

Fig. 8. Part of an open colour base.

by the two spikes again. This way we obtain an *open colour base*, and the recreated spikes are called *free*, see Fig. 8.

The open colour bases have the following properties due to Lemma 14:

- An open colour base is colourable using four colours, i.e. the edge set of a colour base is partitionable in four aC_4fms.
- All these four-colourings are equivalent modulo changing colours, i.e., a colouring is uniquely determined by the colours of four spikes having pairwise distinct colours.

Now we are able to describe the whole reduction graph. Consider a boolean formula F in conjunctive normal form as inputs of **SATISFIABILITY**. Let $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ be of clauses $C_j = L_j^1 \vee \cdots \vee L_j^i$ in the variables $x_1, x_2, \ldots, x_n$. Each literal L_j^k is either a variable x_i or a negated variable $\overline{x_i}$. We restrict the inputs F of **SATISFIABILITY** to be boolean formulas in conjunctive normal form with

- each clause contains two or three literals,
- each variable appears exactly twice positive and
- each variable appears exactly once negative.

SATISFIABILITY remains NP-complete under this restrictions, see [21]. This implies $2n \leqslant 2m \leqslant 3n$, i.e., F consists of $3m - 3n$ clauses of two literals and $3n - 2m$

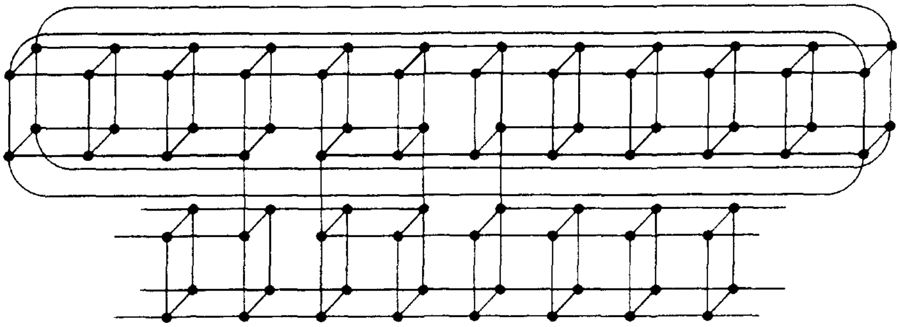

Fig. 9. Connecting the colour base with a truth assignment component.

clauses of three literals. For instance, we have $j_i = 2$ for $1 \leqslant i \leqslant 3(m-n)$ and $j_i = 3$ for $3(m-n) < i < n$.

We create the following components:

- m satisfaction test components (one for each clause). Next we create
- n truth assignment components (one for each variable). We colour the component coresponding to x_i by four colours, namely 'x_i', '$\overline{x_i}$' and two additional colours. Further we create
- a colour base consisting of $4n$ spiked cubes. We colour this component by 'true', 'false' and the two additional colours of the truth assignment components.

Clearly the colouring of a component is done such that all edges are coloured and each colour class is an aC_4fm.

Now we connect these components. First we choose an independent set M of edges in the colour base. M is choosen such that the colour base opened at M has

- m pairs of free spikes coloured 'true',
- $3(m-n)$ pairs of free spikes coloured 'false' and
- n pairs of free spikes of both the additional colours.

Second for each variable we choose in the corresponding truth assignment component an independent set M_i, $1 \leqslant i \leqslant n$, of edges, two coloured 'x_i' and one for each other colour and open the components at M_i. We identify (with respect to the bipartition) one pair of free spikes coloured with an additional colour from each truth assignment component with one pair of free spikes having the same colour from the colour base, see Fig. 9.

Next we identify (with respect to the bipartition) one pair of opposite spikes from each satisfaction test component with a pair of free spikes coloured 'true' from the colour base, see Fig. 10. This does not creates any C_4 between the components.

Last we link the pair of free spikes coloured '$\overline{x_i}$' from the truth assignment component corresponding to x_i to one pair of opposite spikes from the satisfaction test component corresponding to that clause containing the literal $\overline{x_i}$, that is, for both colour classes, we identify these endpoints of (free) spikes having degree one. Note, that each

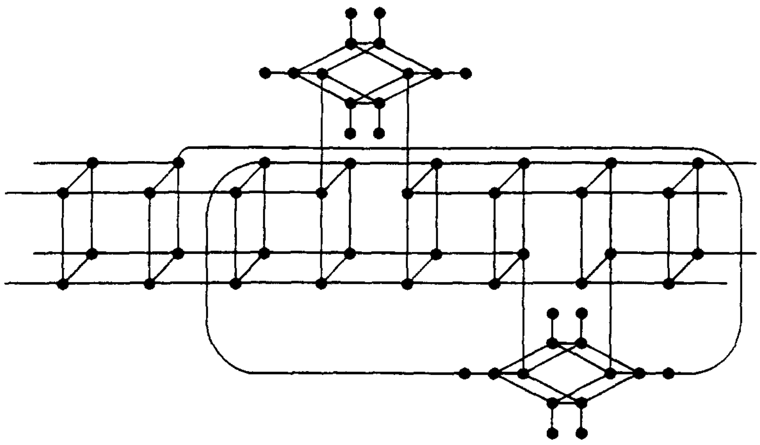

Fig. 10. Connecting the colour base with two satisfaction test componets.

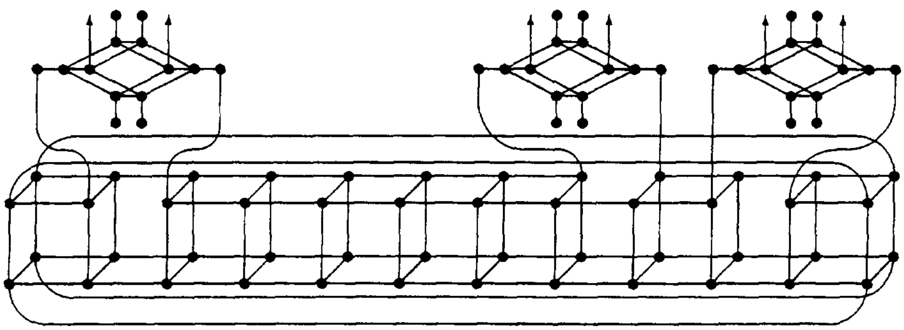

Fig. 11. connecting a truth assignment component with satisfaction test components

pair of spikes has endpoints of degree one in both colour classes. Hence this linking respects the bipartition and does not create a C_4, see Fig. 11.

Analogously, we link the pairs of free spikes coloured 'x_i' to pairs of opposite spikes from the satisfaction test components representing clauses containing the literal x_i. The remaining pairs of opposite spikes in satisfaction test components (corresponding to clauses with two literals) are linked to the remaining pairs of free spikes coloured 'false' in the colour base, see Fig. 10.

Forgetting about the colourings of the components completes the construction of the reduction graph. Observe that all components are bipartite graphs. All connections between different components are made with respect to the bipartition. Each C_4 of the reduction graph is contained in one component.

Theorem 4. *MC_4 restricted to bipartite graphs is NP-complete.*

Proof. It is easy to see that MC_4 belongs to NP. Simply, guess the partition of edge set of the input graph and check deteministicly whether each class of the partition is an aC_4fm.

Now we show that the reduction graph corresponding to a formula F (as described above) is colourable with four colours such that each colour class is an aC_4fm if F is satisfiable. First we fix a satisfying thruth assignment of $x_1, x_2, \dots, x_n$. We remember the colouring of the truth assignment components and the colour base mentioned in the construction. In the truth assignment component of x_i we replace colour 'x_i' by 'true' and colour '$\overline{x_i}$' by 'false' if x_i is true. If x_i is false, we use 'false' instead of 'x_i' and 'true' instead of '$\overline{x_i}$'. We are done with this part of the proof by extending this partial edge colouring to a total edge colouring which describes a partition of the edge set of the reduction graph into four aC_4fms.

Therefore, we consider a satisfaction test component. Two opposite spikes are already coloured 'true' since identified with free spikes from the colour base. We extend this to maximal aC_4fm in the component. The three remaining pairs of opposite spikes are linked to spikes coloured 'true' or 'false'. We choose one pair of opposite spikes linked to spikes coloured 'true'. This is possible since the corresponding clause is satisfied. This pair of opposite spikes is coloured 'false' and again we extend this to maximal aC_4fm in the component. Now it is easy to colour the remaining edges in the component using two additional colours.

It remains to show that a partition of the edge set of our reduction graph into four aC_4fms implies that the formula F is satisfiable. We choose such a partition and colour the edges of the graphs with respect to this partition. For the free spikes of the colour base we choose colour 'true', if identified with spikes in satisfaction test components and colour 'false' if linked to spikes in satisfaction test components. We choose two additional colours for the remaining aC_4fms. For instance, all edges created by identifying a free spike of the colour base and a free spike of a truth assignment component receive one of the additional colours. Hence, by construction, at the five pairs of free spikes of a truth assignment component each colour appears at least once, and either 'true' or 'false' appear at two pairs. All spikes linked to a spike of a satisfaction test component are coloured either 'true' or 'false'.

We define a truth assignment for $x_1, \dots, x_n$. The boolean variable x_i is true, if two pairs of free spikes are coloured 'true', otherwise false. We consider an unsatisfied clause of the formula F. The corresponding satisfaction test component has one pair of opposite spikes coloured 'true' since identified with free spikes of the colour base, and all the other spikes are linked to spikes coloured 'false' since the claus is not satisfied. This implies that colour 'false' is not used inside the component contradicting the partition into four aC_4fms. Hence all clauses are satisfied. This completes the proof. □

Claim 2. *MC_5 is NP-complete.*

Sketch of the proof. We consider the icosahedron graph, which admits a nice covering by five independent sets of edges. A lemma analogous to Lemma 14 is easy to show for the icosahedron graph, even if some edges are opened to form a pair of spikes. In a proof of the claim these graphs can play the role of the spiked cube in reduction above.

8. The edge-clique cover problem

In this section we consider problem [GT18] of [12], called here

$$\mathrm{ECC} = \{(G,k)\colon G \text{ is a graph and } \kappa_e(G) \leqslant k\}.$$

This problem restricted to bipartite graphs was shown to be NP-complete by Orlin, see [26]. We prove here the NP-completeness for the class of chordal bipartite graphs. The reduction is from VERTEX COVER, [GT1] in [12], first to the problem

$$\mathrm{BCC} = \{(G,F,k)\colon G = (V,E) \text{ is a graph, } F \subseteq E \text{ and } \kappa_e(G,F) \leqslant k\}$$

and from this problem to ECC. For the subclass of bipartite distance hereditary graphs ECC is solvable in polynomial time.

8.1. Chordal bipartite graphs and bipartite distance hereditary graphs

In this section we give the definitions of the graph classes considered and list some well-known properties.

Definition 10. A bipartite graph $G = (X,Y,E)$ is called *chordal bipartite graph* (or $(6,1)$-chordal bipartite graph) if each cycle of length at least six has a chord. G is called *bipartite distance hereditary graph* (or $(6,2)$-chordal bipartite graph) if each cycle of length at least six has two chords. An edge $\{x,y\}$ of G is *bisimplicial* if $G[N(x) \cup N(y)]$ is a complete bipartite graph.

The assertions of the following lemmas are easy to see or proved in [13,2].

Lemma 15. *Every chordal bipartite graph with at least one edge has a bisimplicial edge. A bipartite graph (X,Y,E) with bisimplicial edge $\{x,y\}$ is chordal bipartite if and only if $(X,Y,E \setminus \{\{x,y\}\})$ is chordal bipartite.*

Lemma 16. *A chordal bipartite graph is bipartite distance hereditary if and only if it does not contain a domino (see Fig. 12). Bipartite distance hereditary graphs are the hull of K_1 under* PV *and* FT *operations (see Definition 7).*

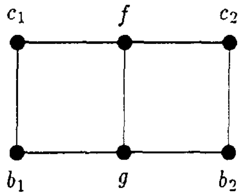

Fig. 12. Domino.

8.2. First reduction

The first reduction is of the component design type. We have two types of components, one corresponds to the vertices of a graph and the other to the edges of this graph. The vertex components G_i allow two standard coverings of distinct cardinality. The bigger cover has the advantage to be extendable to cover the edge components (corresponding to incident edges) too. We will use the bigger type T to cover components corresponding to vertices in the vertex cover of the original graph, while using the smaller type R otherwise.

Now consider an arbitrary graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. We create the following components.

- For each i, $1 \leqslant i \leqslant n$, we take $2m+1$ copies of the C_4. We denote these connected components by $G_{i,l} = (\{a_{i,l}, b_{i,l}, c_{i,l}, d_{i,l}\}, \{\{a_{i,l}, b_{i,l}\}, \{b_{i,l}, c_{i,l}\}, \{c_{i,l}, d_{i,l}\}, \{d_{i,l}, a_{i,l}\}\})$, $1 \leqslant l \leqslant 2m+1$.
- For each k, $1 \leqslant k \leqslant m$ we take a copy $(\{f_k, g_k\}, \{\{f_k, g_k\}\})$ of the K_2.

We connect these components in the following way:

1. If $e_k = \{v_i, v_j\}$ we link $G_{i,2k}, G_{j,2k}$ and the vertices f_k and g_k to obtain the component H_k. H_k contains all edges of $K_{5,5}$ except $\{b_{i,2k}, c_{j,2k}\}$ and $\{b_{j,2k}, c_{i,2k}\}$. That is, we can get H_k from the graph shown in Fig. 13 by four false twin operations (Definition 7).
2. For all i, $1 \leqslant i \leqslant n$ and all l, $1 \leqslant l \leqslant 2m$ we identify the vertices $d_{i,l}$ and $a_{i,l+1}$. So we obtain G_i as a chain of concatenated components $G_{i,1}, G_{i,2}, \ldots G_{i,2m+1}$, see Fig. 14.
3. We add edges $A = \{\{a_{i,k}, a_{j,l}\} \colon 1 \leqslant i,j \leqslant n \wedge 1 \leqslant k,l \leqslant 2m+2 \wedge a_{i,k} \neq a_{j,l} \wedge k+l \equiv 1 \pmod 2\}$ with $a_{i,2m+2} = d_{i,2m+1}$. So the a's and d's induce a complete bipartite subgraph.

This completes the construction of the reduction graph H corresponding to our first reduction.

Lemma 17. *For all graphs G the graph H described above is a chordal bipartite graph.*

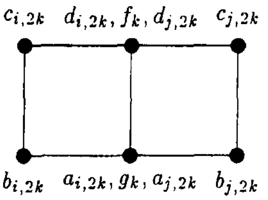

Fig. 13. FT-reduction of the component H_k representing the edge $e_k = \{v_i, v_j\}$.

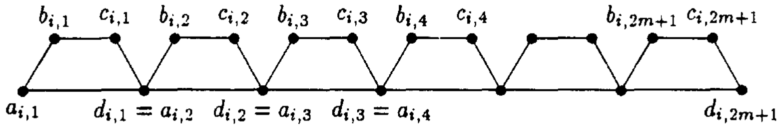

Fig. 14. Component G_i representing the vertex v_i.

Proof. It is easy to see that the components G_i and H_k are bipartite. The connection is made with respect to the bipartition. Here we give a bipartition of $V(H)$:

$$X = \{a_{i,l}, b_{i,l+1}, c_{i,l}, d_{i,l+1} \colon 1 \leqslant i \leqslant n \wedge l \equiv 0 \pmod 2\} \cup \{g_k \colon 1 \leqslant k \leqslant m\},$$
$$Y = \{a_{i,l}, b_{i,l+1}, c_{i,l}, d_{i,l+1} \colon 1 \leqslant i \leqslant n \wedge l \equiv 1 \pmod 2\} \cup \{f_k \colon 1 \leqslant k \leqslant m\}.$$

We show that $H = (X, Y, E)$ is chordal bipartite using Lemma 15. Therefore, observe that each edge in $D = \{\{b_{i,j}, c_{i,j}\} : 1 \leqslant i \leqslant n \wedge 1 \leqslant j \leqslant 2m+1\}$ is bisimplicial in (X, Y, E), each edge in $C = \bigcup\{I(b_{i,j}) \cup I(c_{i,j}) \colon 1 \leqslant i \leqslant n \wedge 1 \leqslant j \leqslant 2m+1\} \setminus D$ is bisimplicial in $(X, Y, E \setminus D)$, each edge in $B = \{\{f_k, g_k\} : 1 \leqslant k \leqslant m\}$ is bisimplicial in $(X, Y, E \setminus (C \cup D))$, each edge in A is bisimplicial in $(X, Y, E \setminus (B \cup C \cup D))$, and $A \cup B \cup C \cup D = E$. □

Now we describe the edge set $F \subset E(H)$ that we use in the reduction from VERTEX COVER to BCC. We take

$$\begin{aligned} F = \; & \{\{d_{i,1}, c_{i,1}\} \colon 1 \leqslant i \leqslant n\} \cup \{\{a_{i,l}, b_{i,l}\}, \{d_{i,l}, c_{i,l}\} \colon 1 \leqslant i \leqslant n \text{ and } 2 \leqslant l \leqslant 2m\} \\ & \cup \{\{a_{i,2m+1}, b_{i,2m+1}\} \colon 1 \leqslant i \leqslant n\} \cup \{\{f_k, g_k\} \colon 1 \leqslant k \leqslant m\}. \end{aligned}$$

Suppose $\mathscr{B}$ is a family of F-maximal edge-cliques in H covering F, i.e., for every $B_1 \in \mathscr{B}$ a biclique B_2 of H with $B_1 \cap F \subset B_2 \cap F$ does not exist. We have the following types of F-maximal bicliques B.

- B is of type R if there exist indices i and l such that $1 \leqslant i \leqslant n$, $1 \leqslant l \leqslant 2m+1$ and $B \cap F = \{\{d_{i,l-1}, c_{i,l-1}\}, \{a_{i,l}, b_{i,l}\}\}$.
- B is of type S if there exist indices i, j and k such that $1 \leqslant i, j \leqslant n$, $1 \leqslant k \leqslant m$, $e_k = \{v_i, v_j\}$ and either $B \cap F = \{\{a_{i,2k}, b_{i,2k}\}, \{f_k, g_k\}, \{a_{j,2k}, b_{j,2k}\}\}$ or $B \cap F = \{\{d_{i,2k}, c_{i,2k}\}, \{f_k, g_k\}, \{d_{j,2k}, c_{j,2k}\}\}$.

- B is of type T if there exist indices i, j and k such that $1 \leqslant i, j \leqslant n$, $1 \leqslant k \leqslant m$, $e_k = \{v_i, v_j\}$ and $B \cap F = \{\{d_{i,2k}, c_{i,2k}\}, \{d_{i,2k}, c_{i,2k}\}, \{f_k, g_k\}\}$ or there exist indices i, j and l such that $1 \leqslant i, j \leqslant n$, $1 \leqslant l \leqslant 2m$ and $B \cap F = \{\{a_{i,l}, b_{i,l}\}, \{d_{i,l}, b_{i,l}\}\}$.

Claim 3. *Every F-maximal biclique in H has exactly one of the types* R, S *or* T.

Proof of the claim. The maximal bicliques of the domino in Fig. 12 are $I(f)$, $I(g)$, $\{\{f,g\},\{g,b_1\},\{b_1,c_1\},\{c_1,f\}\}$ and $\{\{f,g\},\{g,b_2\},\{b_2,c_2\},\{c_2,f\}\}$. H_k has the same number of maximal bicliques because it is created from the domino by FT operations. This is useful since every biclique of H containing $\{f_k, g_k\}$ is a biclique of H_k.

Let B be a F-maximal biclique of H. First let $\{f_k, g_k\} \in B$ for some k. If additionally $I(f_k) \subseteq B$ or $I(g_k) \subseteq B$ then B is of type S. Otherwise we have $E(G_{i,l}) \subseteq B$ (for some i and l) and B is of type T.

Now let $\{f_k, g_k\} \notin B$ for all k. Again, if additionally $E(G_{i,l}) \subseteq B$ (for some i and l) and B is of type T. Otherwise $I(a_{i,l}) \subseteq B$ (for some i and l) and B is of type R. □

Let $\mathscr{B}$ be a family of bicliques of H. We define

$$|\mathscr{B}|_i = \sum_{B \in \mathscr{B}} \frac{\min(|B \cap F \cap E(G_i)|, 1)}{|\{j : B \cap F \cap E(G_j) \neq \emptyset\}|},$$

$$\mathscr{R}_i = \{I(a_{i,l}) \colon 1 \leqslant l \leqslant 2m\},$$

$$\mathscr{T}_i = \{E(G_{i,l}) : 1 \leqslant l \leqslant 2m+1 \text{ and if } l = 2k \text{ then } v_i \notin e_k\}$$
$$\cup \{E(H_k - \{b_{j,2k}, c_{j,2k}\}) \colon 1 \leqslant k \leqslant m \text{ and } e_k = \{v_i, v_j\}\}.$$

Note that all bicliques in $\mathscr{R}_i$ have type R and all bicliques in $\mathscr{T}_i$ have type T. The following claim is easy to verify.

Claim 4. *Let $\mathscr{B}$ be a family of F-maximal bicliques in H covering F. Then*

- $|\mathscr{B}| = \sum_{i=1}^{n} |\mathscr{B}|_i$,
- $\bigwedge_{i=1}^{n} |\mathscr{B}|_i \geqslant 2m$,
- $|\mathscr{T}_i|_i = 2m + 1$, *and*
- $|\mathscr{B}|_i = 2m \iff \{B \cap E(G_i) : B \in \mathscr{B}\} = \mathscr{R}_i \cup \{\emptyset\}$.

Theorem 5. *The problem BCC remains NP-complete when restricted to chordal bipartite graphs.*

Proof. Of course the problem is in NP. We reduce from to the NP-complete problem VERTEX COVER, [GT1] of [12]:

$$\text{VERTEX COVER} = \{(G,k) \colon G = (V,E) \text{ is a graph, } S \subseteq V, |S| \leqslant k$$
$$\text{and for all } e \in E \text{ holds } e \cap S \neq \emptyset\}.$$

First we show $(G,k) \in$VERTEX COVER implies $\kappa_e(H,F) \leqslant 2nm + k$ for the reduction graph constructed above, $n = |V(G)|$ and $m = |E(G)|$.

We consider a set $S \subset V$ witnessing $(G,k) \in$ VERTEX COVER. We define the following family of edge-cliques in H:

$$\mathscr{B} = \{B: B \in \mathscr{T}_i \wedge v_i \in S\} \cup \{B: B \in \mathscr{R}_i \wedge v_i \notin S\}.$$

By Claim 4 we have $|\mathscr{B}| = 2nm + |S|$. $\mathscr{B}$ covers $F \subset E(H)$ since for all $e \in E(G)$ holds $e \cap S \neq \emptyset$. Suppose, on contrary, $\{f_k, g_k\}$ is not covered by an edge-clique in $\mathscr{B}$. Then neither $\mathscr{T}_i$ nor $\mathscr{T}_j$ is a subset of $\mathscr{B}$ if $e_k = \{v_i, v_j\}$. Hence $e_k \cap S = \emptyset$ — a contradiction.

Now we show $\kappa_e(H,F) \leqslant 2nm + k$ implies $(G,k) \in$VERTEX COVER.

Suppose $\mathscr{B}$ is a family of F-maximal edge-cliques in H covering F. First we may suppose that an edge $\{f_k, g_k\}$ is covered by at most one biclique of type S in $\mathscr{B}$. Otherwise we are able to replace two such bicliques by the pair of bicliques of type T covering $\{f_k, g_k\}$.

We define a modified F-cover $\mathscr{B}'$ by

$$\mathscr{B}' = \{B: B \in \mathscr{T}_i \wedge |\mathscr{B}|_i \geqslant 2m + 1\} \cup \{B: B \in \mathscr{R}_i \wedge |\mathscr{B}|_i = 2m\}.$$

By Claim 4 we have $|\mathscr{B}| \geqslant |\mathscr{B}'|$. Next we show that $\mathscr{B}'$ covers F. Suppose, on the contrary, this is not true. Then there is an edge $e_k = \{v_i, v_j\}$ of G such that $\{f_k, g_k\} \in F$ is not covered by $\mathscr{B}'$. By construction of $\mathscr{B}'$ we deduce $|\mathscr{B}|_i = |\mathscr{B}|_j = 2m$ and by Claim 4 we have $\{B \cap E(G_i) : B \in \mathscr{B}\} = \mathscr{R}_i \cup \{\emptyset\}$. Now claim 3 implies that $\mathscr{B}$ does not cover $\{f_k, g_k\}$ — a contradiction.

Finally, for $S = \{v_i : |\mathscr{B}|_i \geqslant 2m + 1\}$ and all $e \in E(G)$ holds $e \cap S \neq \emptyset$ since $\mathscr{B}'$ is exactly the F-cover we would define for S at the beginning of this proof. □

8.3. Second reduction

Now we start to describe the reduction from BCC to ECC. This construction was used by Orlin in [26] too. Suppose $G = (X, Y, E)$ is a bipartite graph and $F \subseteq E$. We add an ear to each edge in $E \setminus F$ in the following way. For $\{x, y\}$ we create new vertices (xy) and (yx) and add the edges $\{x, (xy)\}, \{(xy), (yx)\}$ and $\{(yx), y\}$. So we contain the reduction graph $G^+ = (X^+, Y^+, E^+)$ with

$$X^+ = X \cup \{(yx): \{x, y\} \in E \setminus F\},$$

$$Y^+ = Y \cup \{(xy): \{x, y\} \in E \setminus F\},$$

$$E^+ = E \cup \{\{x, (xy)\}, \{(xy), (yx)\}, \{(yx), y\}: \{x, y\} \in E \setminus F\}.$$

Lemma 18. *Let G be a chordal bipartite graph. Then G^+ is a chordal bipartite graph too.*

Proof. Clearly, G^+ is bipartite. Suppose C is a cycle of length at least six in G^+ but not in G. Then there is an edge $\{x,y\}$ of G such that C uses the edge $\{(xy),(yx)\}$ in G^+. Hence $\{x,y\}$ is a chord of C. □

Theorem 6. *ECC remains NP-complete when restricted to chordal bipartite graphs.*

Proof. Clearly ECC $\in$ NP. First we show $\kappa_e(G,F) \leqslant k$ implies $(G^+, k+|E\setminus F|) \in$ ECC.

Suppose $\mathscr{B}$ is a family of edge-cliques in G that covers F. We define

$$\mathscr{B}^+ = \mathscr{B} \cup \{\{x,(xy)\},\{(xy),(yx)\},\{(yx),y\},\{x,y\} \colon \{x,y\} \in E\setminus F\}.$$

Then $\mathscr{B}^+$ is a family of edge-cliques in G^+ that covers E^+ and $|\mathscr{B}^+| = |\mathscr{B}| + |E\setminus F|$.

Now we show $(G^+, k+|E\setminus F|) \in$ ECC implies $\kappa_e(G,F) \leqslant k$. For $\{x,y\} \in E\setminus F$ the edge $\{(xy),(yx)\}$ is bisimplicial in G^+. The set of edges of this subgraph is the unique maximal edge-clique that contains $\{(xy),(yx)\}$. So without loss of generality $\mathscr{A} = \{\{x,(xy)\},\{(xy),(yx)\},\{(yx),y\},\{x,y\} \colon \{x,y\} \in E\setminus F\} \subseteq \mathscr{B}^+$. By the construction of G^+ there is no edge of F covered by an edge-clique of $\mathscr{A}$. Hence $\mathscr{B} = \mathscr{B}^+ \setminus \mathscr{A}$ covers F and $|\mathscr{B}| = |\mathscr{B}^+| - |E\setminus F|$. This completes the proof. □

8.4. Efficient algorithm

In contrast to the NP-completeness for chordal bipartite graphs we are able to present a polynomial-time algorithm computing κ_e for a more restricted class here. If the input is chosen as an arbitrary bipartite graph, in some cases the algorithm cannot terminate, since step $(*)$ is not executable. But if the algorithm terminates, the result is correct whenever the input has been a bipartite graph.

The following algorithm computes $\kappa_e(G)$ for bipartite distance hereditary graphs G. It is based on a simple fact: a bisimplicial edge is contained in exactly one maximal edge-clique, see Section 8.1.

```
input: a bipartite distance hereditary graph G = (V,E).
begin
        ecc := 0;
        F := E;
        remove all vertices isolated in (V,F) from V;
        while V ≠ ∅ do
        begin
(*)             choose a bisimplicial edge e of G[V] with e ∈ F;
                ecc := ecc + 1;
                remove all edges dependent on e from F;
                remove all vertices isolated in (V,F) from V;
        end;
end.
output: ecc, the cardinality of a minimum edge-clique cover of G.
```

Suppose we are able to choose a bisimplicial edge e of $G[V]$ with $e \in F$ in each stage of the iteration. Then one can prove in a straightforward manner the correctness of the algorithm. Now suppose $\{x, y\}$ is an edge in $E \setminus F$ in some stage of the iteration. Then there was a prior stage with a bisimplicial edge $\{a, b\}$ dependent on $\{x, y\}$. In this stage a, b, x, y induce a C_4 in (V, E) since all edges incident with a or b disappear after removing a and b. Since x and y are not removed in this stage they possess neighbours c and d, respectively. If $\{x, y\}$ is bisimplicial in this stage, then $\{c, d\}$ is an edge in E and so a, b, c, d, x and y induce a domino in G contradicting the fact that G is a bipartite distance hereditary graph. Otherwise in later stages both x and y are removed since all incident edges are removed from F.

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