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MATHEMATICS

CONTENTMENT IN GRAPH THEORY: COVERING GRAPHS WITH CLIQUES

BY

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ABSTRACT

Fundamental questions posed by Boole in 1868 on the theory of sets have in recent years been translated to problems in graph theory. The major problems that this paper deals with are determining the minimum number of complete subgraphs of graph G which include all of the edges of G, and determining the minimum number of complete bipartite subgraphs which cover G. The two problems are of a very similar nature. Determining whether there is a projective plane of order p is a special case of the former problem. The latter problem has a natural translation into matrix theory which yields tight upper and lower bounds. An elementary proof is given for Graham's theorem. Two non-obvious classes are given for which the above problems are easily handled; however, this author doubts that these classes can be extended significantly. Two new problems are shown in this paper to be NP-complete. Finally, several conjectures and unsolved problems are posed within the body of the paper.

Section 1: INTRODUCTION

Unless stated otherwise, a graph G will contain no loops or multiple edges. If vertex v is adjacent to vertex w, we will denote it as the symmetric relation $v \to w$. (This notation will be used with a different meaning in the case of digraphs.) The edge joining vertices v and w will be denoted as the unordered pair (v, w).

A complete graph is a graph in which every pair of vertices is adjacent. A complete subgraph of graph G will be called a clique. A clique is not necessarily a maximal complete subgraph. A clique will sometimes be written as an unordered set of vertices. A clique is called trivial if it consists of only one vertex.

The set of vertices of G is denoted as $V(G)$. If H is a subgraph of G then the set of vertices of H is $V(H)$.

If X is a set of vertices of G then the subgraph of G induced by X is the subgraph consisting of the vertices in X and all edges of G with both endpoints in X. It will be denoted as $G|X$. Similarly if H is a subgraph of G then $H|X$ will denote the subgraph of G with vertices in $V(H) \cap X$ and all edges in H with both endpoints in X.

Definition: The *content* of graph G is equal to the minimum number of edge disjoint cliques of G whose union includes all of the edges of G. The content of G will be denoted as $C(G)$.

Definition: The *R-content* of graph G is equal to the minimum number of cliques of G (with edge repetitions allowed) whose union includes all of the edges of G. The R-content of G will be denoted as $RC(G)$.

Definition: A set S of k edge disjoint cliques is called a *content decomposition* of G if $k=C(G)$ and S includes all of the edges of G.

Definition: A set S of k cliques is called an *R-content covering* if $k=RC(G)$ and if S includes all the edges of G.

The motivation for the definitions of content and R-content stem from problems in set theory.

Given a family of sets $S_1, S_2, \ldots, S_n$ it is possible to associate with it a graph G with vertices $x_1, x_2, \ldots, x_n$ such that the number of edges joining x_i to x_j is equal to $|S_i \cap S_j|$.

Conversely, E. Szpilrajn-Marczewski in [14] proved the following theorem:

Theorem 1.1. Let G be a graph with possibly multiple edges and with vertices $x_1, \ldots, x_n$. Then there exists a set S and a family of subsets $S_1, \ldots, S_n$ of S such that $|S_i \cap S_j|$ is equal to the number of edges in G joining x_i to x_j.

A problem posed by Erdös, Goodman, and Posa in [4] and motivated by Boole in [1] is the following: what is the minimum number of elements in a set S that satisfies Theorem 1.1? Boole referred to the number as the content of the system of sets. It is no coincidence that the content of the system of sets is equal to the content of the graph. It stems from the fact that an element of S induces a clique in G.

Similarly the following theorem is true:

Theorem 1.2. Let G be a graph with no multiple edges and with vertices $x_1, \ldots, x_n$. Then there exists a set S and a family of subsets $S_1, \ldots, S_n$ of S such that $|S_i \cap S_j| \geqslant 1$ iff x_i and x_j are adjacent.

The minimum number of elements in the set S is equal to the R-content of G. Upper bounds for the content and R-content were given by Erdös, Goodman and Posa in [4], by M. Hall Jr. in [5], and by Lovasz in [12]. The problem was also considered by Ryser in [13].

Section 2: some elementary remarks on the content and R-content

Remark 2.1: If G is a graph and W is a subset of the set of vertices of G, then $C(G) \geqslant C(G|W)$ and also $RC(G) \geqslant RC(G|W)$.

PROOF. Let $S=\{C_1, C_2, \ldots, C_k\}$ be a content decomposition (or an R-content covering) of G. Let $S^*=S|W=\{C_1|W, \ldots, C_k|W\}$. It follows that S^* is a decomposition (or a covering) of $G|W$.

REMARK 2.2: Suppose G is a graph with the property that every edge belongs to a unique maximal clique. Then $C(G)=$number of distinct maximal cliques. Furthermore, G has a unique content decomposition.

PROOF. Let C_α be the unique maximal clique that contains edge α. Then it follows that if C is any clique containing edge α, $C \subseteq C_\alpha$.

Suppose that $S=\{C_1, \ldots, C_k\}$ is a content decomposition of G. Suppose further that for some edge α, $C_\alpha \notin S$. Then choose index set I minimal such that $C_\alpha \subseteq \bigcup_{i \in I} C_i$. For each $i \in I$ C_i has an edge of C_α and thus by the above $C_i \subseteq C_\alpha$. Thus $C_\alpha = \bigcup_{i \in I} C_i$ and replacing the C_i for $i \in I$ by C_α we have a decomposition with fewer cliques. This contradicts the assumption that S is a content decomposition. Thus the remark is true.

REMARK 2.3: Without loss of generality we may add the restriction that each clique in an R-content covering be maximal.

PROOF. Follows directly from definition of R-content.

REMARK 2.4: Suppose that in graph G edge α belongs to a unique maximal clique C_α. Then C_α occurs in every R-content covering of G (with the restriction mentioned in Remark 2.3).

PROOF. Edge α must occur in some clique. By Remark 2.3, C_α is the only choice.

REMARK 2.5: Suppose G has a subset A of edges with the property that every edge $\alpha \in A$ belongs to a unique maximal clique C_α and that $S=\{C_\alpha : \alpha \in A\}$ is a covering of G. Then S is the unique R-Content covering of G.

PROOF. Follows directly from Remark 2.4.

REMARK 2.6: Let G be a graph. Let G_w be the graph induced by removing vertex w. Suppose $RC(G)=RC(G_w)$. Suppose also that G_w has a unique R-content covering. Then it follows that G has a unique R-content covering.

PROOF. Suppose that $S_1=\{C_1, \ldots, C_r\}$ and $S_2=\{D_1, \ldots, D_r\}$ are two distinct R-content coverings of G. Let $Z=V(G)-\{w\}$. Then $S_1|Z$ and $S_2|Z$ both contain only maximal cliques with respect to Z and cover G_w. Thus by hypothesis $S_1|Z=S_2|Z=$the unique R-content covering of G_w. Furthermore since all cliques in S_1 (and S_2) are maximal it is easy to reconstruct S_1 (and S_2) from $S_1|Z$ (and $S_2|Z$) by adjoining vertex w to all cliques possible. However, since the reconstruction is the same for S_1 and S_2 we must have that $S_1=S_2$.

NOTE: The extension of Remark 2.6 to content and unique content decompositions is not true as can be seen in the following example.

EXAMPLE 1: Let G be the graph described in fig. 1. Then G_w has a unique content decomposition. $C(G_w) = C(G) = 4$. However,

$$S_1 = \{(v_1, v_2, w), (v_3, v_4, w), (v_1, v_4), (v_2, v_3)\}$$
$$S_2 = \{(v_1, v_2), (v_3, v_4), (v_1, v_4, w), (v_2, v_3, w)\}$$

are two distinct content decompositions.

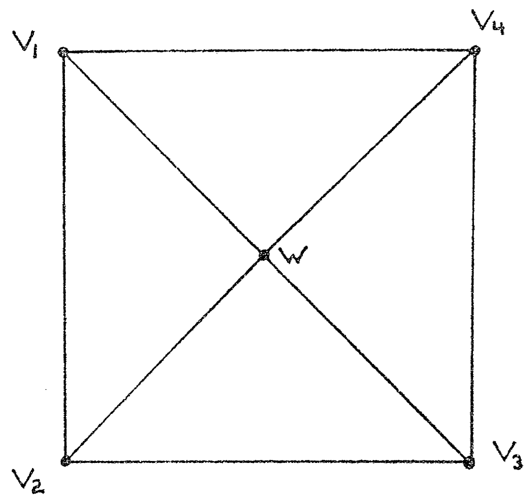


Figure 1

Section 3: THE DEBRUIJN-ERDÖS THEOREM

THEOREM: (DeBruijn-Erdös) Let $G = K_n$ = the complete graph with n vertices. Let $S = \{C_1, \ldots, C_r\}$ be a decomposition of G with $r \leqslant n$. Then one of the following is true.

1) $r = 1$; $C_1 = G$ [Type 1]
2) $r = n$; one of the cliques contains $n-1$ vertices. The remaining $n-1$ cliques are edges. [Type 2]
3) $r = n$ and all the cliques are of size $k+1$ where $k^2 + k + 1 = n$. [Type 3]

PROOF. See [2] for alternate formulation and proof.

REMARK 3.1: It is not difficult to show that a decomposition of type 3 corresponds to a projective plane. The cliques represent lines and the vertices represent points (or vice-versa.)

COROLLARY 3.2: Let $G = K_{p^2+p+1} - K_{p+1}$. Then $C(G) = p^2 + p$ iff there w exists a projective plane of order p.

Proof. Suppose first that $C(G) \leqslant p^2+p$. Then let $S=\{C_1, \ldots, C_r\}$ be a content decomposition of G. Then there exists clique C_{r+1} with $p+1$ vertices such that $S=\{C_1, \ldots, C_{r+1}\}$ is a decomposition of K_{p^2+p+1} into $\leqslant p^2+p+1$ cliques. By DeBruijn-Erdös the only possibility is that the decomposition is of type 3 and there is a projective plane of order p. In this case,

$$C(G)=p^2+p.$$

Conversely assume there is a projective plane of order p. Then there is a type 3 decomposition of K_{p^2+p+1} into cliques of size $p+1$. It follows that there is a decomposition of G into p^2+p cliques. Thus $C(G) \leqslant p^2+p$. By the above $C(G)=p^2+p$.

Corollary 3.3: Let $G=K_n-K_2$. Then G has a unique content decomposition consisting of $n-2$ edges and one clique with $n-1$ vertices.

Proof. Analogous to the proof of Corollary 3.2.

Definition: Let T_{2n} = the n-dimensional octohedron $=K_{2n}-(v_1, v_2)-$ $-(v_3, v_4)-\ldots-(v_{2n-1}, v_{2n})$ = complement of a perfect matching.

Proposition 3.4: $C(T_{2n}) \geqslant n+1$ for $n \geqslant 2$.

Proof. Denote the vertices of T_{2n} as $v_1, v_2, \ldots, v_n, w_1, \ldots, w_n$. Let $Y=\{v_1, \ldots, v_n\}$. Let $Z=\{w_1, \ldots, w_n\}$. For all $i \neq j$ we have that $v_i \rightarrow v_j$, $w_i \rightarrow w_j$ and $v_i \rightarrow w_j$. Let $S=\{C_1, \ldots, C_r\}$ be a content decomposition of T_{2n}. Assume that $r \leqslant n$.

Thus $S|Y$ and $S|Z$ both form decompositions of K_n into $\leqslant n$ cliques. By DeBruijn-Erdös the decompositions are of three possible types.

Case 1. Either $S|Y$ or $S|Z$ is of type 1. Without loss of generality assume that it is $S|Y$ and that $C_1=(v_1, v_2, \ldots, v_n)$. Then edges (v_1, w_2), (v_2, w_3), $\ldots$, (v_{n-1}, w_n) and (v_n, w_1) occur in different cliques in S. Thus S has $\geqslant n+1$ cliques contrary to assumption.

Case 2. Either $S|Y$ or $S|Z$ is of type 2. Without loss of generality assume $S|Y$ is of type 2 and that $C_1=(v_1, v_2, \ldots, v_{n-1}, w_n)$ or else that $C_1=(v_1, \ldots, v_{n-1})$. In both cases C_1 does not include any edge of $G|Z$. Thus the other $n-1$ cliques contain all the edges of $G|Z$. But by DeBruijn-Erdös and case 1 this is impossible. Thus neither $S|Y$ nor $S|Z$ is of type 2.

Case 3. Both $S|Y$ and $S|Z$ are projective plane decompositions of K_n. Thus the number of vertices in each C_i is $2r+2$ where $r^2+r+1=n$. Then the number of edges in all the cliques of S is $n\binom{2r+2}{2}$. But we know that the number of edges of G is $\binom{2n}{2}-n$. By equating the two formulae we get that $r=0$ and $n=1$. Thus for $n \geqslant 2$ the theorem holds.

Note: The inequality is sharp in the sense that $C(T_6)=4$. A content decomposition is

$$S=\{(v_1, v_2, w_3), (v_1, v_3, w_2), (w_1, v_2, v_3), (w_1, w_2, w_3)\}$$

Conjecture 1:

$$\operatorname*{limit}_{n\to\infty} \frac{C(T_{2n})}{n}=1$$

The inequality for $RC(T_{2n})$ is reversed for $n\geqslant 3$; i.e. $RC(T_{2n})\leqslant n+1$. This will follow from the following lemma.

Lemma 3.5: $RC(T_{2n})\leqslant RC(T_{2n-2})+1$.

Proof. Using the same notation as in the proof of proposition 3.4 we see that each maximal clique contains either vertex v_i or vertex w_i but not both. Hence each maximal clique contains exactly n vertices. Let $S=\{C_1, C_2, \dots, C_r\}$ be an R-content covering of T_{2n-2}. Since for each i, v_i and w_i are symmetric, it is then possible to relabel vertices so that $C_1=(v_1, v_2, \dots, v_{n-1})$. Let $S^*=\{D_1, \dots, D_{r+1}\}$ where for $i=2, \dots, r$ $D_i=$ $=C_i \cup \{w_n\}$; $D_1=(v_1, v_2, \dots, v_n)$; $D_{r+1}=(w_1, w_2, \dots, w_{n-1}, v_n)$. It follows that S^* is a covering of T_{2n}. Thus the lemma is true.

Corollary 3.6: $RC(T_{2n})\leqslant n+1$ for $n\geqslant 3$.

Proof. $RC(T_6)=4$; the corollary then follows from induction based on Lemma 3.5.

Unsolved Problem: What is the asymptotic behavior of $RC(T_{2n})$? I make the following remark without proof. For explanation of the terms see [7].

Remark 3.7: Let $S=$ class of Boolean functions $f\colon \{0, 1\}^n \to \{0, 1\}$ with the property that all of the prime implicants have length n. Let S' be the subclass of S consisting of functions with the additional property that for every assignment of two variables the resulting function is not identically 0. Let f' be the function of S' with the fewest number of prime implicants. Then the number of implicants of f is exactly $RC(T_{2n})$.

Sketch of Proof: Label the vertices of T_{2n} as $x_1, x_2, x_3, \dots, x_n$, $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$. Then each clique of T_{2n} is associated uniquely with a conjunction of the n boolean variables.

Section 4: The content and R-content of the line graph

Definition: Let G be a graph. The line graph of G which will be denoted as G^* is defined as follows: the vertices of G^* correspond in a 1:1 manner with the edges of G; two vertices are adjacent in G^* if the corresponding edges of G are both incident with a common vertex of G.

For examples, see fig. 5 and fig. 6. The following four remarks are straight forward and will be presented without proof.

REMARK 4.1: If $v \in V(G)$ has valence $\geqslant 2$ then v induces a clique in G^*.

REMARK 4.2: Let $C_1, C_2, \ldots, C_n$ be the cliques of G^* induced by the vertices $v_1, v_2, \ldots, v_n$ of G. These cliques (some of which may be trivial) form a decomposition of G^*.

Let $V_2(G)$ = set of vertices of G of degree 2 or more.

REMARK 4.3: $C(G^*) \leqslant |V_2(G)|$

REMARK 4.4: Let $G_\alpha = G$ with edge α removed. Then $C(G_\alpha^*) \leqslant C(G^*)$.

THEOREM 4.5: If $G \neq K_3$ and if G is connected then $C(G^*) = |V_2(G)|$.

PROOF. By Remark 4.3, I need only prove that $C(G^*) \geqslant |V_2(G)|$. The theorem holds for the G described in fig. 2. $C(G^*) = V_2(G) = 3$.

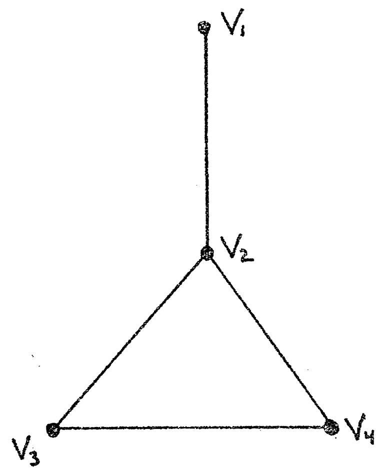


Figure 2

Assume that the theorem is false. Let G be a counter example of the theorem such that G^* is minimal with respect to number of edges.

REDUCTION 1: G does not have two adjacent vertices of valence $\neq 2$.

PROOF OF REDUCTION 1: Assume false. Let α be the edge of G joining the two vertices. Then

(1) $$C(G_\alpha^*) = |V_2(G_\alpha)|$$

by induction hypothesis.

(2) $$|V_2(G_\alpha)| = |V_2(G)|$$

by hypothesis.

$$(3) \qquad |V_2(G)| \geqslant C(G^*).$$

By Remark 4.3 and

$$(4) \qquad C(G^*) \geqslant C(G^*_\alpha)$$

by Remark 4.4.

Thus combining the above equations we get that $C(G^*) = |V_2(G)|$ contrary to assumption.

Reduction 2: Suppose vertex v of G is adjacent only to vertices x and w in G. Then x is adjacent to w.

Proof of Reduction 2: Assume false. Let α_1 be the vertex of G^* corresponding to (v, x) and let α_2 be the vertex of G^* corresponding to (v, w). Then in G^* the clique (α_1, α_2) is maximal and must occur in any decomposition. Create a new graph $\bar{G}$ from G by replacing vertex v with two vertices v_1, v_2 of degree 1, with v_1 adjacent to x, and v_2 adjacent to w. Then $\bar{G}^* = G^*$ with edge (α_1, α_2) removed. Then

$$C(\bar{G}^*) = |V_2(\bar{G})|$$

by induction hypothesis.

$$|V_2(\bar{G})| = |V_2(G)| - 1$$

by hypothesis.

$$C(\bar{G}^*) = C(G^*) - 1$$

because (α_1, α_2) accounts for a clique of G^* but no clique for $\bar{G}^*$.

Thus by the above equations $C(G^*) = |V_2(G)|$ contrary to assumption.

Reduction 3: G must be the graph consisting of n distinct 3-cycles joined at an articulation point, for some $n \geqslant 2$. (See fig. 3.)

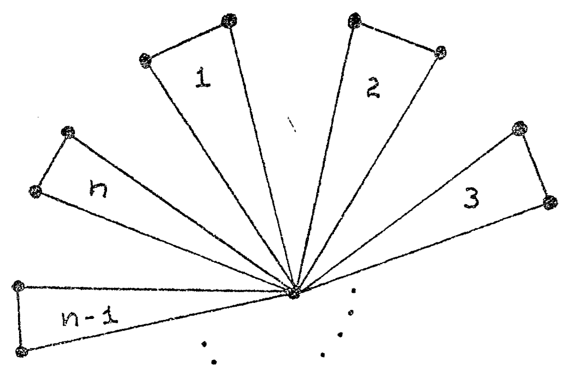


Figure 3

PROOF OF REDUCTION 3: By Reduction 1 every vertex of G that is not of degree 2 is adjacent to a vertex of degree 2. By Reduction 2 every vertex of degree 2 is contained in a 3-cycle. By Reduction 1, at most 1 vertex of any 3-cycle is of degree $\geqslant 3$. By hypothesis G is not a K_3. Thus the graph described above is the only graph possible.

THE FINAL CASE: It will now be shown that the theorem is even true for the graph of reduction 3. Let us write the vertices of G^* as $v_1, \ldots, v_{2n}, w_1, w_2, \ldots, w_n$ where the v's correspond to the edges of G which meet at the articulation point while the w's correspond to the remaining edges. Thus the following are all the maximal cliques of G^*: $(v_1, v_2, \ldots, v_{2n})$, (v_1, v_2, w_1), (v_3, v_4, w_2), $(v_5, v_6, w_3) \ldots (v_{2n-1}, v_{2n}, w_n)$. Thus in any decomposition of G^* there are two possibilities for edge (w_j, v_{2j}); it either appears in a clique consisting of only two vertices (Type 1) or it appears in a clique of three vertices (Type 2).

Suppose S is a decomposition of G^* into $\leqslant 2n$ edge disjoint cliques. Let $m =$ number of cliques of type 2. Then $2(n-m) =$ number of cliques of type 1. Thus the rest of the graph is covered with $\leqslant 2n-(2n-2m)-$ $-m=m$ cliques. The remaining graph is G' which is equal to K_{2n} with a matching of m edges removed.

If $m=1$ there cannot be a covering by Corollary 3.3. If $m>1$ there exists an induced subgraph of G' which is isomorphic to T_{2m}. Thus by Proposition 3.4 and Remark 2.1 there cannot be a decomposition of G' with $\leqslant m$ cliques. Thus $C(G^*) \geqslant 2n+1 = |V_2(G)|$ and the theorem is true.

DEFINITION: A 3-*wing* is a 3-cycle of a graph with the additional property that two of the three vertices have degree 2 while the third vertex has degree 3.

REMARK 4.6: If G possesses a 3-wing then G^* does not have unique content decomposition.

PROOF. If the vertices corresponding to the 3-wing in G are labeled $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ it can easily be seen that there are two choices for $G^*|\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. The cliques are either (α_1, α_2), (α_1, α_3), $(\alpha_2, \alpha_3, \alpha_4)$ or else $(\alpha_1, \alpha_2, \alpha_3)$, (α_2, α_4) (α_3, α_4). See fig. 4.

REMARK 4.7: If G is the graph described in fig. 5 then G^* does not have a unique content decomposition.

PROOF. By Theorem 4.5 $C(G^*)=7$. The decompositions are:

$$S_1 = \{(w_1, v_1, v_2), (w_2, v_3, v_4), (w_3, v_5, v_6), (v_1, v_3, v_5), (v_1, v_4, v_6), (v_2, v_3, v_6), (v_2, v_4, v_5)\}$$

and

$$S_2 = \{(w_1, v_1), (w_1, v_2), (w_2, v_3), (w_2, v_4), (w_3, v_5) (w_3, v_6), (v_1, v_2, v_3, v_4, v_5, v_6)\}$$

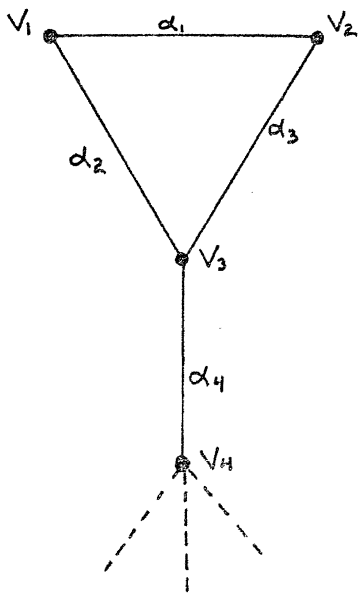


Figure 4

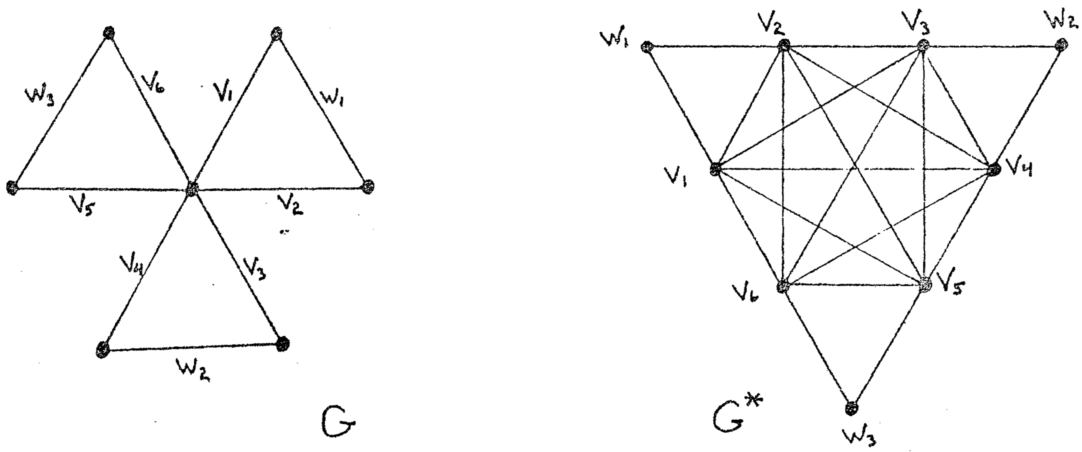


Figure 5

Remark 4.8: If G does not have a 3-wing and if G is not the graph described in Remark 4.7, then G^* has a unique content decomposition and it is the decomposition induced by $V_2(G)$.

Proof. The proof is exhaustive in nature and too long to be worth including in this paper.

Definition: *A wing* is a subgraph of G which is a 3-cycle with the additional property that exactly two of the three vertices has degree 2.

THEOREM 4.9: Suppose $G \neq K_3$ and that G is connected. Let $m =$ number of distinct wings of G. Then

$$RC(G^*) = |V_2(G)| - m.$$

PROOF. Reductions 1 and 2 of the proof of theorem 4.5 carry through as before. The R-content of the graph G in Reduction 3 is $n+1$ because there exists $n+1$ edges of G^* which belong to distinct unique maximal cliques which include all edges of G^*.

COROLLARY 4.10: Let $m_1 =$ number of distinct wings of G; let $m_2 =$ number of components of G that are isolated 3-cycles. (i.e. components equivalent to K_3). Then

$$C(G^*) = |V_2(G)| - 2m_2$$
$$RC(G^*) = |V_2(G)| - 2m_2 - m_1.$$

PROOF. This follows directly from Theorems 4.5 and 4.9 using induction on number of components.

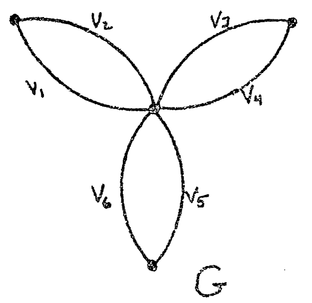
REMARK 4.11: G^* always possesses a unique R-content covering.

PROOF. Recall that all cliques in the covering must be maximal. Assume there is a minimum counter example. Reductions 1 and 2 of the proof of theorem 4.5 again carry through. The graph described in Reduction 3 also possesses a unique R-content covering by Remark 2.5.

REMARK 4.12: The generalization of Theorem 4.5 to multigraphs is not true.

PROOF. In fig. 6 $|V_2(G)| = 4$ while $C(G) = 3$. The decomposition is

$$S = \{(v_1, v_2, v_3, v_4), (v_1, v_2, v_5, v_6), (v_3, v_4, v_5, v_6)\}.$$



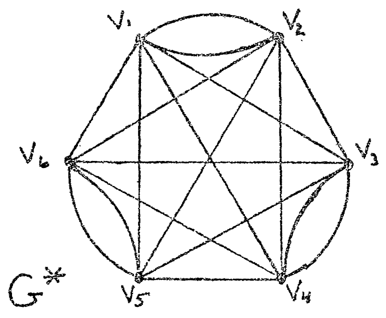


Figure 6

Section 5: THE BICONTENT AND R-BICONTENT

A bipartite graph G is a graph in which the vertices may be partitioned into two disjoint sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, y_3, \ldots, y_n\}$ with the property that for all i, j we have that $x_i \not\leftrightarrow x_j$ and $y_i \not\leftrightarrow y_j$.

A complete bipartite graph is a bipartite graph for which for all $i=1, \ldots, m$ and for all $j=1, \ldots, n$ we have that $x_i \to y_j$.

The *bicontent* of bipartite graph G is equal to the minimum number of edge disjoint complete bipartite subgraphs whose union includes all of the edges of G. The bicontent of G will be denoted as $B(G)$. If $k=B(G)$ we will say that set S is a *bicontent decomposition* of G if S contains k edge disjoint complete bipartite subgraphs whose union includes all of the edges of G.

The *R-bicontent* of bipartite graph G is equal to the minimum number of complete bipartite subgraphs (with edge repetitions allowed) whose union includes all of the edges of G. The R-bicontent of G will be denoted as $RB(G)$. If $k=RB(G)$ we will say that set S is an *R-bicontent covering* if S contains k complete bipartite subgraphs whose union includes all the edges of G.

From here on a complete bipartite subgraph will be abbreviated as a CBS.

The following remarks are analogous to the remarks in section 2. Their proofs will be omitted. Unless stated otherwise, all graphs G that are referred to in this section will be bipartite graphs with vertex set $X \cup Y$ with X and Y as above.

REMARK 5.1: If G is a graph and W is a set of vertices of G, then $B(G) \geqslant B(G|W)$.

REMARK 5.2: Suppose that G is a graph with the property that every edge belongs to a unique maximal CBS. Then $B(G)$ is equal to the number of distinct maximal CBS's. Furthermore, G has a unique bicontent decomposition.

REMARK 5.3: We may assume without loss of generality that each CBS in an R-bicontent covering is maximal.

REMARK 5.4: Suppose that in graph G edge α belongs to a unique maximal CBS called C_α. Then C_α occurs in every R-bicontent covering.

REMARK 5.5: Suppose that G has a subset A of edges with the property that every edge $\alpha \in A$ belongs to a unique maximal CBS C_α and that $S=\{C_\alpha : \alpha \in A\}$ is an R-bicontent covering of G. Then S is the unique R-bicontent covering of G.

REMARK 5.6: Let G_w be the graph induced from G by deleting vertex w. Suppose that $RB(G)=RB(G_w)$. Suppose further that G_w has a unique R-bicontent covering. It then follows that G has a unique R-bicontent covering.

Section 6: TRANSLATIONS OF RB(G) AND B(G) INTO PROPERTIES OF (0, 1) MATRICES

Let $\mathscr{M}=$ class of all $m \times n$ matrices with coefficients in $\{0, 1\}$. For

$A=(a_{ij})$ and $B=(b_{ij})$ with $A, B \in \mathscr{M}$ we write $A \leqslant B$ if for all $i=1, \ldots, m$ and for all $j=1, \ldots, n$ it is true that $a_{ij} \leqslant b_{ij}$.

$A+B$ denotes the usual matrix addition while $A \vee B$ denotes boolean addition. Thus if $C=(c_{ij})=A \vee B$ then $c_{ij}=1$ if a_{ij} or $b_{ij}=1$; $c_{ij}=0$ if both a_{ij} and b_{ij} are equal to 0.

We will let $r(A)$ denote the rank of matrix A and $t(A)$ will denote the term rank of matrix A. The term rank is equal to the maximum number of 1's no two of which are on a line.

We associate with bipartite graph G a matrix $M_G=M_G(i,j)$ where $M_G(i,j)=1$ if $x_i \to y_j$; $M_G(i,j)=0$ otherwise. Furthermore, we associate with every subgraph H of G a matrix $M_H=M_H(i,j)$ where $M_H(i,j)=1$ if $x \to y$ in H. $M_H(i,j)=0$ otherwise. Clearly $M_H \leqslant M_G$ for all subgraphs H. The following three remarks are easy to prove, and their proofs will be omitted.

Remark 6.1: Let H be a *CBS* of G. Then $r(M_H)=1$.

Remark 6.2: If $A \in \mathscr{M}$ is such that $A \leqslant M_G$ then there is some subgraph H of G such that $A=M_H$.

Remark 6.3: If $A \in \mathscr{M}$ is such that $A \leqslant M_G$ and $r(A)=1$ then there is a *CBS* H of G such that $M_H=A$.

Remark 6.4: $B(G)$ is equal to the minimum number of rank 1 matrices in $\mathscr{M}$ whose sum is M_G.

Proof. By Remarks 6.1 and 6.3 it follows that *CBS*'s in G correspond in a $1-1$ manner with the rank 1 matrices A in with the property that $A \leqslant M_G$. It is also true that subgraphs H and K of G are disjoint iff $M_K+M_H \leqslant M_G$. Then the remark follows from the definition of bicontent.

Theorem 6.5: $B(G) \geqslant r(M_G)$.

Proof. Let A, B, and C be matrices such that $C=A+B$. It is a theorem of matrix theory that $r(C) \leqslant r(A)+r(B)$. From this and Remark 6.4 the theorem follows.

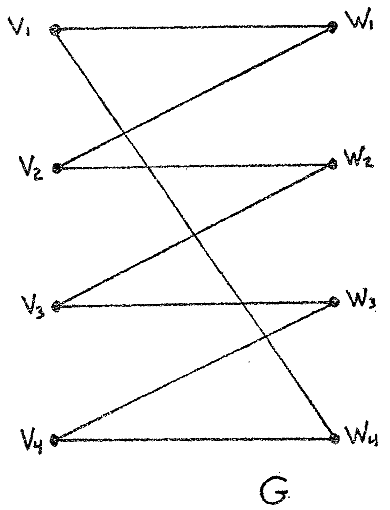
Theorem 6.6: $t(M_G) \geqslant B(G) \geqslant RB(G)$.

Proof. It follows from the definitions that $RB(G) \leqslant B(G)$. Thus it is only necessary to show that $t(M_G) \geqslant B(G)$. For a vertex w of G define span (w) to be the subgraph of G consisting of w and the vertices of G that are adjacent to w and all of the connecting edges. If $H=$ span (w) it follows that all of the 1's in M_H lie in a single line; that is, they all lie in either one row or in one column.

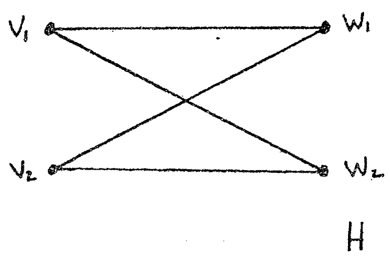
It is a theorem of König that for any $A \in \mathscr{M}$, $t(A)$ is equal to the minimum number of lines of A which include all of the 1's. Translating to graph theory, König's theorem says that $t(M_G)$ is equal to the minimum number of spans of G which include all of the edges of G.

Since a span is a kind of *CBS* as is any subgraph of a span, it follows from Remark 6.4 that $t(M_G) \geqslant B(G)$.

The following examples show that the bounds in Theorems 6.5 and 6.6 are in fact tight. Note that in fig. 7 it is true that $B(G) = RB(G) = t(M_G) = 4$, while $r(M_G) = 3$. Also $B(H) = RB(H) = r(M_H) = 1$, while $t(M_H) = 2$.



$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} M_G$$



$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} M_H$$

Figure 7

REMARK 6.7: $RB(G)$ is equal to the minimum number of rank 1 matrices in $\mathscr{M}$ whose boolean sum is equal to M_G.

PROOF. The proof is analogous at that of remark 6.4.

Section 7: COVERING GRAPHS AND DIGRAPHS AND A NEW PROOF OF GRAHAM'S THEOREM

Let D be a digraph (directed graph) with no loops or multiple edges. If there is an edge directed from x_i to x_j we will write $x_i \to x_j$. The edge will be denoted as the ordered pair (x_i, x_j).

A directed complete bipartite subgraph H of D is a subgraph H such that $V(H) = \{x_1, \ldots, x_r, y_1, \ldots, y_s\}$ with the property that for all i and for all j, $x_i \not\to x_j$; $x_i \to y_j$; $y_i \not\to x_j$; $y_i \not\to y_j$; a directed complete bipartite subgraph will be denoted as a $DCBS$.

Thus it is possible to define the bicontent of D to be the minimum number of edge disjoint $DCBS$'s whose union includes all of the edges of D. Similarly define the R-bicontent of D to be the minimum number of $DCBS$'s whose union includes all of the edges of D. We will use the notation $B(D)$ and $RB(D)$ as before.

REMARK 7.1: It is possible to associate in an obvious manner with any digraph D a bipartite graph G such that $B(D) = B(G_D)$ and $RB(D) = RB(G_D)$.

PROOF. Let $M_D = M_D(i, j)$ where $M_D(i, j) = 1$ if $x_i \to x_j$ and 0 otherwise. From M_D it is possible to create a bipartite graph G_D such that $M_{G_D} = M_D$.

Furthermore, it is clear from the construction that the CBS's of G correspond in a $1-1$ manner with the $DCBS$'s of D.

If G is an ordinary graph, we will denote the adjacency matrix of G as $\bar{M}_G$. Thus $\bar{M}_G(i, j) = 1$ iff vertex v_i is adjacent to vertex v_j. We can extend the definition of B, so that $B(G)$ is the minimum number of edge disjoint CBS's that cover the edges of G.

Given $n \times n$ real matrix A, let $N^+(A)$ and $N^-(A)$ denote respectively the number of positive eigenvalues of A and the number of negative eigenvalues of A. If A is symmetric, then all eigenvalues are real and A has full geometric multiplicity. Hence, in this case $N^+(A) + N^-(A) =$ rank (A). The following theorem is due to H. S. Witsenhausen but appears in [6].

THEOREM 7.2: $B(G) \geqslant \max(N^+(\bar{M}_G), N^-(\bar{M}_G))$.

LEMMA 7.3: Let A be a real symmetric matrix. Let V^+ and V^- denote respectively the vector space generated by the eigenvectors corresponding to positive eigenvalues and the space generated by the eigenvectors with negative eigenvalue. If $v \in V^+$ and $v \neq 0$ then $v^T A v > 0$. If $v \in V^-$ and $v \neq 0$, then $v^T A v < 0$.

PROOF OF LEMMA: Let $\{\lambda_1, \ldots, \lambda_k\}$ denote the distinct positive eigenvalues. If $v \in V^+$ then there exist $\alpha_1, \alpha_2, \ldots, \alpha_k$ and vectors $x_1, \ldots, x_k$ such that $Ax_j = \lambda_j x_j$ and such that $v = \sum_{j=1}^k \alpha_j x_j$. For $i \neq j$, $(x_i^T A) x_j = \lambda_i x_i^T x_j$ while $x_i^T (A X_j) = \lambda_j x_i^T x_j$. Since $\lambda_i \neq \lambda_j$ it follows that $x_i^T A x_j = 0$. Hence, $v^T A v = \sum_{j=1}^t \lambda_i (\alpha_j x_j)^T (\alpha_j x_j) > 0$. A similar argument shows that for $v \in V^-$ with $v \neq 0$, $v^T A v < 0$.

LEMMA 7.4: Let C be a real $n \times n$ matrix. Let $A = C + C^T$. Then $r(C) \geqslant \max(N^+(A), N^-(A))$, where $r(C)$ denotes the rank of C.

PROOF. Let $V^* = \{x\colon Cx = 0; x \in \mathbb{R}^n\}$. If $x \in V^*$ then $x^T A x = x^T (Cx) + (x^T C^T) x = 0$. Hence $x \notin V^+$, $x \notin V^-$ where V^+, V^- are defined as in

Lemma 7.3. Since $V^* \cap V^+ = V^* \cap V^- = \{0\}$ it follows that $\dim V^* + \max(\dim V^+, \dim V^-) \leqslant n$. Thus $(n - r(C)) + \max(N^+(A), N^-(A)) \leqslant n$. Thus $r(C) \geqslant \max(N^+(A), N^-(A))$.

PROOF OF THEOREM 7.2: If G is a graph, let $P(G)$ be the set of digraphs which are orientations of G. Note that $D \in P(G)$ iff $M_D + M_D^T = \overline{M}_G$. If $D \in P(G)$ then any $DCBS$ of D induces a unique CBS of G. Also, given any edge disjoint set of CBS's of G, it is possible to orient the edges of G to get a digraph $D \in P(G)$ such that these CBS's of G correspond to $DCBS$'s of D. It follows that

$$B(G) = \min_{D \in P(G)} B(D).$$

By Theorem 6.5 as applied to digraphs

$$B(D) \geqslant r(M_D).$$

By Lemma 7.4, for all $D \in P(G)$,

$$r(M_D) \geqslant \max(N^+(\overline{M}_G), N^-(\overline{M}_G)).$$

Thus the theorem is true.

COROLLARY 7.5 (Graham) *). $B(K_n) = n - 1$.

PROOF. It is easy to cover the complete graph with $n-1$ edge disjoint CBS's. It is also easy to show that $N^-(\overline{M}_{K_n}) = n - 1$.

To form the *line digraph* D^* of D we associate with each edge (x_i, x_j) of D a vertex A_{ij} of D^*. There is an edge directed from A_{jk} to A_{rs} in D^* if $k = r$.

A *through vertex* of D is a vertex with at least one edge directed in and at least one edge directed out. Let $T(D)$ be the number of distinct through vertices in D.

THEOREM 7.2: $B(D^*) = RB(D^*) = T(D)$.

PROOF. According to a theorem of Harary and Norman in [8] D has the property that every edge belongs to a unique maximal $DCBS$, and that this $DCBS$ is induced by a through vertex. Thus Theorem 7.2 follows from this and a translation of Remark 5.2 to directed graphs.

Section 8: NP-COMPLETENESS

There has been much research in recent years on a class of problems called NP-complete. The reader is referred to [3] and [9] for the exact definitions and further information on NP-completeness.

Briefly and inprecisely a problem P is said to be reducible to another problem Q if it is possible to translate problem P in a polynomial amount of time to problem Q. If P is reducible to Q we will denote it as $P \propto Q$. Problems P and Q are said to be equivalent if $P \propto Q$ and $Q \propto P$.

A problem is said to be NP-complete if it is equivalent to a host of

*) For alternate proof see [6].

combinatorial (and non-combinatorial problems) including the traveling salesman problem, determining the chromatic number of a graph, and determining the maximum size independent set of vertices in a graph. The ideas involved in NP-completeness will be made more clear in the following theorem and its proof.

THEOREM 8.1: The following five problems are all NP-complete.

$P0$) Determine the fewest number of cliques which include all of the vertices of graph G.

$P1$) Determine the fewest number of CBS's of G which will cover a specified subset H of edges of G (where G is bipartite.)

$P2$) Determine $RB(G)$.

$P3$) Determine $RC(G)$ for graph G.

$P4$) Determine the minimum number of cliques of G that cover a specified subset of edges of G.

PROOF. $P0$ is proven to be NP-complete in [7]. Solving $P0$ is exactly the same as determining the chromatic number of the complement of G. For the rest of the proof it will be shown that $P0 \propto P1 \propto P2 \propto P3 \propto P4 \propto P0$.

$P0 \propto P1$. Let G be a graph with vertices $v_1, \ldots, v_n$. Let G' be a graph with vertices $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. In G', $x_i \to y_i$ for all $i=1$ to n. Also $x_i \to y_j$ in G' if $v_i \to v_j$ in G. The specified set H' of edges in G' are the edges (x_i, y_i) for all $i=1$ to n.

Any clique C in G which includes v_i induces a CBS in G' which includes (x_i, y_i). Conversely let C' be a CBS in G' which includes $(x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}), \ldots, \ldots, (x_{j_k}, y_{j_k})$. Then in G it is true that $(v_{j_1}, v_{j_2}, \ldots, v_{j_k})$ is a clique. Thus the minimum number of cliques that cover the vertices in G is equal to the minimum number of CBS's in G' that cover the edges of H'.

$P1 \propto P2$. Let G be a bipartite graph with vertices $x_1, \ldots, x_s, y_1, \ldots, y_t$ and a specified subset H of edges. Let G' be another bipartite graph such that G is a subgraph of G'. In addition, for every edge $(x_i, y_j) \in G$ such that $(x_i, y_j) \notin H$ there are two extra vertices in G' labeled x_{ij} and y_{ij}. In G' $x_{ij} \to y_{ij}$, $x_{ij} \to y_j$, and $x_i \to y_{ij}$. Thus in G', (x_{ij}, y_{ij}) is an element of a unique maximal CBS which includes edge (x_i, y_j). By Remark 5.4 we may assume that the CBS occurs in every R-bicontent covering. Since the construction was carried out for every edge $(x_i, y_j) \in H$, it follows that we may assume that all edges not in H have been covered by CBS's. Thus the number of CBS's of G that cover the edges of H is equal to $RB(G') - |H^c|$ where H^c denotes the set of edges of G that are not in H.

$P2 \propto P3$. Let G be a bipartite graph with vertices $x_1, \ldots, x_s, y_1, \ldots, y_t$. Let G' be a graph such that $G \subset G'$. In addition for every i and j such that $i \neq j$ we have in G' that $x_i \to x_j$ and $y_i \to y_j$. Finally two extra vertices are in G'. Vertex z_1 is adjacent to every x_i. Vertex z_2 is adjacent to every y_i. In G' edges (z_1, x_1) and (z_2, y_1) belong to unique maximal cliques $(z_1, x_1, \ldots, x_s)$ and $(z_2, y_1, \ldots, y_t)$. The remaining edges in G' correspond to edges in G. Also every CBS in G induces a clique in G'. Thus $RB(G) = RC(G') - 2$.

$P3 \propto P4$. Choose H to be all edges of G.

$P4 \propto P0$. Let G be a graph with vertices $v_1, \ldots, v_n$. Let the specified subset of edges of G be $H = \{e_1, \ldots, e_m\}$. Let G' be another graph with vertices $w_1, \ldots, w_m$ where for $i \neq j$ $w_i \to w_j$ in G' iff e_i and e_j belong to a common clique in G. (i.e. if the 3 or 4 end points of edges e_i and e_j are all adjacent to each other.) A clique $(w_{j_1}, \ldots, w_{j_k})$ in G' corresponds to a clique in G which includes edges $e_{j_1}, \ldots, e_{j_k}$. Thus the minimum number of cliques of G covering the edges of H is equal to the minimum number of cliques of G' covering all the vertices.

Comment. The reader with a knowledge of NP-completeness will realize that a proof that $P4 \in NP$ could be substituted for the last part of the above proof; however, the above proof is perhaps more interesting.

It is commonly believed that if a problem is NP-complete, then there is no polynomially bounded algorithm to solve the problem. Thus to get an answer on a computer, heuristics must be used for larger problems.

It seems intuitively obvious that determining the content or bicontent of a graph is NP-complete. However, the author knows of no proof of the assertion. It is known that both problems are in the class NP.

It was prove in [11] that problem $P3$ (and hence also $P4$) is NP-complete. That publication appeared after the original writing of this paper and was discovered independently. The proofs are similar in nature although far from identical.

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