Perfectness of bipartite graphs

Consider a bipartite graph $G = (V, E)$. We make the following observations:

1. each induced subgraph of G is also a bipartite graph;

2. the complement of each induced subgraph of G is also the complement of a bipartite graph;

3. a bipartite graph with at least one edge has: maximum clique size two, color number two (without any edge, clique size and color number are equal to one), hence a bipartite graph G is perfect;

4. a direct, 'constructive' proof of perfectness of the complement of G given below

Proof of complementary perfectness.
Let $\bar{G} = (U \cup W, F)$, be the complement of G with where U and W are the color classes of G. Then U and W each induce a complete subgraph of $\bar{G}$. Let ω denote the maximum clique size in $\bar{G}$, then we need to show that we can color the vertices of $\bar{G}$ with ω colors.

1. Partition $U = U_1 \cup U_2$ and $W = W_1 \cup W_2$, where $U_2 \cup W_2$ forms the largest clique, of size ω. We need ω distinct colors to color $U_2 \cup W_2$, and now claim that the coloring of $U_2 \cup W_2$ can be extended to a complete coloring without any extra colors.

2. Consider any subset U' of U_1, and let $W' := \{v \in W_2 | \exists u \in U', uv \notin F\}$, that is W' is the set of neighbors of U', in G restricted to $U_1 \cup W_2$. Then each vertex in $W_2 \setminus W'$ must be adjacent to each vertex in U'. As the clique $U_2 \cup W_2$ is of maximum size, $|U'| \leq |W'|$, otherwise exchanging these two would increase the clique size.

3. From matching theory we now know that there is a matching in G of size $|U_1|$, from U_1 to W_2. In other words, for each $u \in U_1$ there is a distinct mate $m(u) \in W_2$ such that $\{u, m(u)\} \in E$, or rather, $\{u, m(u)\} \notin F$. Moreover $m(u) \neq m(u')$, for $u \neq u'$. Give $u \in U_1$ the same color as $m(u)$.

4. Similarly, we can color the vertices in W_1, using the colors of U_2.