Projections

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1 Trigonometry formulary

1.1 Fundamental relations

$$\cos^2 a + \sin^2 x = 1 \quad (1)$$

$$\frac{1}{\cos^2} = 1 + \tan^2 x \quad (2)$$

1.2 Addition formulae

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \quad (3)$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b \quad (4)$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \quad (5)$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b \quad (6)$$

1.3 Double angle formulae

We deduce from the above equations (and Eq. 2) that:

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \quad (7)$$

$$\sin(2x) = 2 \sin x \cos x \quad (8)$$

$$\tan(2x) = \frac{2\tan x}{1 + \tan^2 x} \quad (9)$$

1.4 Semi-angle formulae

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad (10)$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad (11)$$

$$\tan^2 x = \frac{\sin(2x)}{1 + \cos(2x)} = \frac{1 - \cos(2x)}{\sin(2x)} \quad (12)$$

1.5 Trigonometric functions depending on inverse trigonometric functions

$$
\begin{aligned}
\cos(\arccos x) &= x & (13)\\
\cos(\arcsin x) &= \sqrt{1-x^2} & (14)\\
\cos(\arctan x) &= \frac{1}{\sqrt{1+x^2}} & (15)\\
\sin(\arccos x) &= \sqrt{1-x^2} & (16)\\
\sin(\arcsin x) &= x & (17)\\
\sin(\arctan x) &= \frac{x}{\sqrt{1+x^2}} & (18)\\
\tan(\arccos x) &= \frac{\sqrt{1-x^2}}{x} & (19)\\
\tan(\arcsin x) &= \frac{x}{\sqrt{1-x^2}} & (20)\\
\tan(\arctan x) &= x & (21)
\end{aligned}
$$

2 Generalities

A projection is a transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$, $(X, Y) = F(\alpha, \delta)$

$$F : \begin{pmatrix} \alpha \\ \delta \end{pmatrix} \to \begin{pmatrix} X = f_1(\alpha, \delta) \\ Y = f_2(\alpha, \delta) \end{pmatrix}$$

The associated differential transform is

$$\mathrm{d}X\mathrm{d}Y = |\det J_F(\alpha, \delta)|\mathrm{d}\alpha\mathrm{d}\delta \tag{22}$$

with J_F the Jacobian of the transformation F:

$$J_F = \begin{pmatrix} \frac{\partial f_1}{\partial \alpha} & \frac{\partial f_1}{\partial \delta} \\ \frac{\partial f_2}{\partial \alpha} & \frac{\partial f_2}{\partial \delta} \end{pmatrix} \tag{23}$$

A projection is equiareal if elementary surfaces are proportional, so if

$$\mathrm{d}X\mathrm{d}Y \propto \cos\delta\mathrm{d}\alpha\mathrm{d}\delta \tag{24}$$

So if $|\det J_F(\alpha, \delta)| \propto \cos\delta$.

2.1 Changing the center of the projection

By default we consider in this document that the centre of the projection is the vernal point, so the point of Eulidean coordinates $(1, 0, 0)$.

To set the projection cetner to (α_c, δ_c), simpliy mutliply the (x, y, z) coordinate of a celestial point by the rotation matrix:

$$\begin{pmatrix} \cos\alpha_c\cos\delta_c & \sin\alpha_c\cos\delta_c & \sin\delta_c \\ -\sin\alpha_c & \cos\alpha_c & 0 \\ -\cos\alpha_c\sin\delta_c & -\sin\alpha_c\sin\delta_c & \cos\delta_c \end{pmatrix}$$

before the projection.

And by the transpose of the matrix after the deprojection.

3 Zenithal (or azimuthal) projections

Input:

$$x = \cos(\rho) \tag{25}$$
$$y = \sin(\rho)\cos(\theta) \tag{26}$$
$$z = \sin(\rho)\sin(\theta) \tag{27}$$

in which ρ is the angular distance between the reference point $(1, 0, 0)$ and the point to be projected (x, y, z) and θ is the angle between $\vec{y} = (0, \vec{1}, 0)$ and $(0, \vec{y}, z)$, i.e. $\theta = \frac{\pi}{2} - PA$, PA being the position angle (east of north) between the reference point and the point to be projected.

Output:

$$X = R\cos(\theta) \tag{28}$$
$$Y = R\sin(\theta) \tag{29}$$

We have the relations

$$x^2 + y^2 + z^2 = 1 \tag{30}$$
$$\sqrt{y^2 + z^2} = \sin(\rho) = r \tag{31}$$
$$\theta = \arctan\frac{y}{x} \tag{32}$$
$$\theta = \arctan\frac{Y}{X} \tag{33}$$
$$R = \sqrt{X^2 + Y^2} \tag{34}$$

In the FITS reference paper (Calabretta & Greisen, 2002), different conventions are used. They use ϕ_{cg} and θ_{cg} (cg stands for "Calabretta and Greisen") wich are linked to our conventions by the formula:

$$\theta = \frac{\pi}{2} - \frac{\pi}{180}\phi_{cg} \tag{35}$$
$$\rho = \frac{\pi}{2} - \frac{\pi}{180}\theta_{cg} \tag{36}$$

Depending on if we look from the inner of the unit sphere or the outter of the unit sphere, we may also have

$$\theta = \frac{\pi}{2} - \frac{\pi}{180}\phi_{cg} \tag{37}$$

3.1 AZP: Zenithal perspective

The point of projection P is on the x-axis and its coordinates (in the local frame) are $(-\mu, 0, 0)$. The projection plane is not perpendicular to the x-axis but is tilted of an angle γ (angle between the perpendicular to the plane at the point (1, 0, 0) and the x-axis) in the xz plane.

- μ - is the opposite of the abscissa of the projection point P and must be $\neq -1$ (in which case P is on the projection plane)
- γ - angle between the projection plane and the x-axis in the xz plane.

In FITS, μ is provided by keyword PVi_1a (no units, default value = 0) and γ is provided by keyword PVi_2a (in degrees, default value = 0). The projection is defined by the following equations:

$$X = D\cos\theta \tag{38}$$

$$Y = D\frac{\sin\theta}{\cos\gamma} \tag{39}$$

$$D = \frac{(\mu+1)\sin\rho}{(\mu+\cos\rho) - \sin\rho\sin\theta\tan\gamma} \tag{40}$$

Remark: $\mu+1$ is the coordinate of P from the point (1, 0, 0) and allong the opposite of the x-axis.

3.1.1 Projection

We rewrite

$$D = \frac{(\mu+1)\sin\rho}{(\mu+x) - z\tan\gamma} \tag{41}$$

So

$$X = D\frac{y}{\sin\rho} = \frac{y(\mu+1)}{(\mu+x) - z\tan\gamma} \tag{42}$$

$$Y = D\frac{z}{\sin\rho\cos\gamma} = \frac{z(\mu+1)}{\cos\gamma(\mu+x) - z\sin\gamma} \tag{43}$$

The limit of the projection if $|\mu| > 1$ is $x = \frac{-1}{\mu}$, so if $x < \frac{-1}{\mu}$ a point is not projected. From this value, we can compute the maximum value of D, which depends on θ:

$$D_{max}(\theta) = \frac{(\mu+1)\sqrt{1-\frac{1}{\mu^2}}}{(\mu-\frac{1}{\mu}) - \sqrt{1-\frac{1}{\mu^2}}\sin\theta\tan\gamma} \tag{44}$$

$$= \frac{\mu+1}{\sqrt{\mu^2-1} - \sin\theta\tan\gamma} \tag{45}$$

The limit of the projection if $|\mu| < 1$ is more complex. The limit is the plane parallel to the projection plane and containing the point of position $(-\mu, 0, 0)$. We can then write the limit: $(x, y, z).(-\cos\gamma, 0, \sin\gamma) <= \mu\cos\gamma$, which gives

$$(x + \mu)\cos\gamma \leq z\sin\gamma \tag{46}$$

we do not use $\tan\gamma$ not to have to change the inequality according to the sign of $\cos\gamma$ and to be still able to use the formula when $\gamma \approx \pm\pi/2$.

3.1.2 Deprojection

We have to revert D to obtain ρ.

$$D = \sqrt{X^2 + Y^2\cos^2\gamma} \tag{47}$$

$$\sin\theta = \frac{Y\cos\gamma}{D} \tag{48}$$

$$D(\mu + \cos\rho) - \sin\rho Y\sin\gamma = (\mu + 1)\sin\rho \tag{49}$$

First of all, we compute the distance max to be sure the given point are in the limits of the projection:

$$D_{max}(Y) = \frac{\mu + 1}{\sqrt{\mu^2 - 1} - \frac{Y}{D}\sin\gamma} \tag{50}$$

So if $D \leq D_{max}(Y)$ or ,equivalently, if $D\sqrt{\mu^2 + 1} \leq (\mu + 1) + Y\sin\gamma$, we continue. Dividing by D on both sides of Eq. (49), we rewrite it:

$$\frac{(\mu + 1) + Y\sin\gamma}{D}\sin\rho - \cos\rho = \mu \tag{51}$$

If $(\mu + 1) = -Y\sin\gamma$, the equation reduces to $\cos\rho = -\mu$ so $\rho = \arccos(-\mu)$ (which is possible only of $\mu \in [-1, 1]$. To better handle small distance (i.e. $D \approx 0$) we rewrite the previous equation:

$$\sin\rho - \frac{D}{(\mu + 1) + Y\sin\gamma}\cos\rho = \frac{\mu D}{(\mu + 1) + Y\sin\gamma} \tag{52}$$

If $D = 0$, it reduces to $\sin\rho = 0$ so to $\rho = 0$ (can't be π since the opposite point is hidden in htis projection). Previous equation is an equation of the form:

$$A\sin a - B\cos a = C \tag{53}$$

That we solve rewriting it

$$\frac{A}{\sqrt{A^2 + B^2}}\sin a - \frac{B}{\sqrt{A^2 + B^2}}\cos a = \frac{C}{\sqrt{A^2 + B^2}} \tag{54}$$

and using the equality ($A/\sqrt{A^2 + B^2}$ and $B/\sqrt{A^2 + B^2}$ are $\in [0, 1]$ so they can be the value if sines or cosines)

$$\sin(a - b) = \sin a\cos b - \cos a\sin b \tag{55}$$

Leading to

$$a = \arcsin \frac{C}{\sqrt{A^2 + B^2}} + \arctan(B/A) \tag{56}$$

Because of the arcsin function which results are $\in [-\pi/2, \pi/2]$, in the case of $|\mu| < 1$ we may choose the solution

$$a = \pi - \arcsin \frac{C}{\sqrt{A^2 + B^2}} + \arctan(B/A) \tag{57}$$

So in our case in which $C = \mu B$, we have

$$\rho = \arcsin \frac{\mu B}{\sqrt{B^2 + 1}} + \arctan B \tag{58}$$

with

$$B = \frac{D}{(\mu + 1) + Y \sin\gamma} \tag{59}$$

In case of $|\mu| < 1$, we have the solution:

$$\rho = \begin{cases} \arcsin \frac{\mu B}{\sqrt{B^2+1}} + \arctan B & \text{if } \sin(\rho) > 0 \\ \pi - \arcsin \frac{\mu B}{\sqrt{B^2+1}} + \arctan B & \text{if } \sin(\rho) < 0 \end{cases} \tag{60}$$

But we are interested in $sin\rho$ and $\cos\rho$, and trigonometric functions are costly (in term of CPU) so using:

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \tag{61}$$

we can write

$$\begin{aligned} \sin\rho &= \sin\left(\arcsin \frac{\mu B}{\sqrt{B^2 + 1}} + \arctan B\right) && (62) \\ &= \frac{\mu B}{\sqrt{B^2 + 1}} \frac{1}{\sqrt{1 + B^2}} + \sqrt{1 - \frac{\mu^2 B^2}{B^2 + 1}} \frac{B}{\sqrt{1 + B^2}} && (63) \\ &= (w_2 + w_3 B)/w_1 && (64) \end{aligned}$$

and

$$\begin{aligned} \sin((\pi - a) + b) &= \sin(\pi - a)\cos b + \cos(\pi - a)\sin b && (65) \\ &= (\sin\pi\cos a - \cos\pi\sin a)\cos b + (\cos\pi\cos a + \sin\pi\sin a))\sin b && (66) \\ &= \sin a \cos b - \cos a \sin b && (67) \\ &= (w_2 - w_3 B)/w_1 && (68) \end{aligned}$$

and using

$$\cos(a + b) = \cos a \cos b - \sin a \sin b \tag{69}$$

we can write

$$\cos\rho = \cos\left(\arcsin\frac{\mu B}{\sqrt{B^2+1}} + \arctan\sqrt{B^2+1}\right) \tag{70}$$

$$= \sqrt{1-\frac{\mu^2 B^2}{B^2+1}}\frac{1}{\sqrt{1+B^2)}} - \frac{\mu B}{\sqrt{B^2+1}}\frac{B}{\sqrt{1+B^2}} \tag{71}$$

$$= (w_3 - w_2 B)/w_1 \tag{72}$$

and

$$\cos((\pi-a)+b) = \cos(\pi-a)\cos b - \sin(\pi-a)\sin b \tag{73}$$

$$= -(\cos a\cos b + \sin a\cos b) \tag{74}$$

$$= -(w_3 + w_2 B)/w_1 \tag{75}$$

with

$$w_1 = \sqrt{B^2+1} \tag{76}$$

$$w_2 = \frac{\mu B}{w_1} \tag{77}$$

$$w_3 = \sqrt{1-w_2^2} \tag{78}$$

So at the end we have all ingredients to compute

$$x = \cos\rho \tag{79}$$

$$y = \sin\rho\frac{X}{D} \tag{80}$$

$$z = \sin\rho\frac{Y\cos\gamma}{D} \tag{81}$$

3.2 SZP: Slant zenithal perspective

The projection plane is tangential to the sphere in $C = (1, 0, 0)$ and the projection point P is located at position (x_p, y_p, z_p) in the local frame (i.e. the frame centered at the center O of the unit sphere). In FITS (see Calabretta & Greisen, 2002, §5.1.2), the position of P is given by three parameters:

- μ - distance OP (PVi_1a, no units, default = 0), negative if P is on the side of planewards hemisphere, positive otherwise
- ϕ_c - angle (PVi_2a, in degrees, default = 0) between the opposite of the Y-axis and the projection of P onto the XY-plane)
- θ_c - angle (PVi_3a, in degrees, default = 90) between the y-axis rotated of the angle $\frac{\pi}{2} - \frac{\pi}{180}\phi_c$ around the x-axis and the point of intersection between PO and the unit sphere on the planewards side (planewards hemisphere)

In our framework, we prefer using an always positive distance r_p:

- r_p - distance (no units) of the point P to the center of the sphere, always positive

- θ_p - angle (in radian, default = $\pi/2$) between the X-axis and the segment joining C and the projection of P on the XY-plane

- ρ_p - angular distance (in radians, default = 0) between the point C and the intersecition of PO with the unit sphere

So that the coordinates of P are:

$$x_p = r_p \cos\rho_p \tag{82}$$
$$y_p = r_p \sin\rho_p \cos\theta_p \tag{83}$$
$$z_p = r_p \sin\rho_p \sin\theta_p \tag{84}$$

Using longitude l_r and latitude b_r:

$$x_p = r_p \cos b_p \cos l_p \tag{85}$$
$$y_p = r_p \cos b_p \sin l_p \tag{86}$$
$$z_p = r_p \sin b_p \tag{87}$$

so that the relation betwee both systems are:

$$\theta_p = \arctan \frac{\sin b_p}{\cos b_p \sin l_p} \tag{88}$$
$$\rho_p = \arccos \cos b_p \cos l_p \tag{89}$$

The relations between the FITS conventions and this conventions are (see Eq. 35 and 36):

$$r_p = |\mu| \tag{90}$$
$$\theta_p = \frac{\pi}{2} - \frac{\pi}{180}\phi_c \tag{91}$$
$$\rho_p = \begin{cases} \frac{\pi}{2} - \frac{\pi}{180}\theta_c, & \text{if } \mu < 0 \\ \frac{\pi}{2} + \frac{\pi}{180}\theta_c, & \text{otherwise} \end{cases} \tag{92}$$

Again, according to if we look from the inside or the outside of the sphere we may consider

$$\theta_p = \frac{3\pi}{2} - \frac{\pi}{180}\phi_c \tag{93}$$

3.2.1 Projection

We consider a point A of coordinates (x, y, z) on the unit sphere and A' of coordinates $(1, X, Y)$ its projection on the projection plane such that PAA' are on a same straitgh line. We note P' of coordinate $(1, y_p, z_p)$ the projection of P onto the projection plane such that PP' is perpendicular to the projection plane. We also note A'' the projection of A on PP' such that $AA'' \perp PP'$.
If $r_p \in [0, 1]$, i.e. the projection point is inside the unit sphere, we can project all point suchs that $x > x_p$. If $r_p > 1$ and $\rho_r > \pi/2$ ($\Leftrightarrow x_p > 0$), then hidden points are nearest points from P which are in the cone of center P and delimited by the tangents of the unit sphere passing through P. We note T a point of the unit sphere such taht TP is tangantial to the sphere an T' the projection of T onto PO (such that the triangle $PT'T$ is right in T'. Angle $\sin \widehat{OPT} = \frac{1}{r_p}$ and $\widehat{TOP} = \frac{\pi}{2} - \widehat{OPT}$.
Hidden points are defined such that:

$$\frac{\vec{OA}.\vec{OP}}{||\vec{OP}||} > \cos \widehat{TOP} = \sin \widehat{OPT}$$

I.e.

$$xx_p + yy_p + zz_p > 1 \tag{94}$$

If $r_p > 1$ and $\rho_r < \pi/2$ ($\Leftrightarrow x_p < 0$), the result is the opposite: only the nearest point from P which are in the cone of center P and delimited by the tangents of the unit sphere passing through P are projected. Hidden points are defined such that:

$$xx_p + yy_p + zz_p < 1 \tag{95}$$

We can write this using the XOR operator (ˆ), so points are hidden is the following expression is true:

$$\left[(r_p \le 1)\ \&\&\ (x \le x_p)\right]\ ||\ \left[(r_p > 1)\ \&\&\ \left((x_p < 0)\hat{}(xx_p + yy_p + zz_p > 1)\right)\right]$$

We now want to compute the projected positions of valid points considering the xz-plane in a first step. We use the triangle $A'_{xz}P_{xz}P'_{xz}$ and $A_{xz}P_{xz}A''_{xz}$ (we use notation $_{xz}$ to designate the projections on the xz plane).

$$\begin{aligned} \tan \widehat{A'_{xz}P_{xz}P'_{xz}} &= \frac{Y - z_p}{1 - x_p} \\ \tan \widehat{A_{xz}P_{xz}A''_{xz}} &= \frac{z - z_p}{x - x_p} \end{aligned}$$

As the two angle are equals, we easily express Y (and similarly for X):

$$X = (1 - x_p)\frac{y - y_p}{x - x_p} + y_p = \frac{y(1 - x_p) - y_p(1 - x)}{x - x_p} \tag{96}$$

$$Y = (1 - x_p)\frac{z - z_p}{x - x_p} + z_p = \frac{z(1 - x_p) - z_p(1 - x)}{x - x_p} \tag{97}$$

3.2.2 Deprojection

We introduces the following simplifying notations from the previous section:

$$\begin{aligned}
T_X &= \tan \widehat{A'_{xy}P_{xy}P'_{xy}} = \frac{X - y_p}{1 - x_p} \\
T_Y &= \tan \widehat{A'_{xz}P_{xz}P'_{xz}} = \frac{Y - z_p}{1 - x_p}
\end{aligned}$$

We need first to compute x from previous equations Eq. 96 and 97 removing y and z terms remembering that $y^2 + z^2 = 1 - x^2$.

$$\begin{aligned}
y &= T_X(x - x_p) + y_p = xT_X - (T_X x_p - y_p) \\
z &= T_Y(x - x_p) + z_p = xT_Y - (T_Y x_p - z_p)
\end{aligned}$$

We introduce notations:

$$\begin{aligned}
T'_X &= T_X x_p - y_p \\
T'_Y &= T_Y x_p - z_p
\end{aligned}$$

We now take the sum of the squares of both previous equations (in y and z) to obtain an equation of second king in x (remembering that $y^2 + z^2 = 1 - x^2$):

$$x^2(T_X^2 + T_Y^2 + 1) - 2x(T_X T'_X + T_Y T'_Y) + (T'^2_X + T'^2_Y - 1) = 0$$

So

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{98}$$

with

$$\begin{aligned}
a &= T_X^2 + T_Y^2 + 1 \\
b &= -2(T_X T'_X + T_Y T'_Y) \\
c &= T'^2_X + T'^2_Y - 1
\end{aligned}$$

If we divide all three coefficients a, b, and c by 2, we find the same result as in Calabretta & Greisen (2002):

$$x = \frac{-b' \pm \sqrt{b'^2 - ac}}{a}$$

with $b' = b/2$. We keep the highest x (that must be $\in [1, -1]$, i.e. following solution (and we derive easily y and z from the expressions of T_X and T_Y):

$$\begin{aligned}
x &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} && (99) \\
y &= T_X(x - x_p) + y_p && (100) \\
z &= T_Y(x - x_p) + z_p && (101)
\end{aligned}$$

About validity if $r > 1$. Coordiantes of a point in the projection plane are $(1, X, Y)$. The projection of the cone of vertex P ($\vec{p}$), axe PO and angle $\psi = \arcsin(1/r_p)$ (so that it is tangential to the unit sphere) on the projection plane is a conic section (so a circle, an ellipse, a parabola or an hyperbola). All point $\vec{v}$ inside the cone are such that (we perfrom a translation to place the origin of the frame at the vertex of cone):

$$|\frac{(\vec{v}-\vec{p}).(-\vec{p})}{||\vec{v}-\vec{p}|| \times ||-\vec{p}||}| \geq \cos\arcsin\frac{1}{r_p}$$

Thus, using Eq. (14),

$$\begin{aligned}
\frac{|\vec{p}^2 - \vec{v}.\vec{p}|}{r_p\sqrt{\vec{v}^2 + r_p^2 - 2\vec{v}.\vec{p}}} &\geq \sqrt{1-\frac{1}{r_p^2}} \\
\frac{|r_p^2 - \vec{v}.\vec{p}|}{\sqrt{\vec{v}^2 + r_p^2 - 2\vec{v}.\vec{p}}} &\geq \sqrt{r_p^2 - 1}
\end{aligned}$$

or, written explicitly:

$$\frac{\left[x_p(1-x_p) + y_p(X-y_p) + z_p(Y-z_p)\right]^2}{(1-x_p)^2 + (X-y_p)^2 + (Y-z_p)^2} \geq r^2 - 1 \tag{102}$$

Remark: in the projection part, $\vec{v}^2 = 1$, so we could have use instead of Eq. (94) and (95)

$$\frac{|r_p^2 - \vec{v}.\vec{p}|}{\sqrt{1 + r_p^2 - 2\vec{v}.\vec{p}}} \geq \sqrt{r_p^2 - 1}$$

i.e.

$$\frac{\left[x_p(x-x_p) + y_p(y-y_p) + z_p(z-z_p)\right]^2}{(x-x_p)^2 + (y-y_p)^2 + (z-z_p)^2} \geq r^2 - 1 \tag{103}$$

but developping this inequation we find the simple forms Eq. (94) and (95)

3.3 TAN: Gnomonic

The Gnomonic projection is a special case of zenithal perspective projection (§3.1) with $\mu = 0$ and $\gamma = 0$.

3.3.1 Projection

We simply apply Eq. (28) and (29) with

$$R = \tan\rho \tag{104}$$

We project the hemisphere such that $\rho \leq \pi/2$ on the full XY-plane with divergences for $\rho = \pi/2$, thus for $x = 0$.

Noting that

$$R = \frac{\sin\rho}{\cos\rho} = \frac{\sin\rho}{x}, \tag{105}$$

and using Eq. (28) and (29) we obtain the projection formulea:

$$X = \frac{y}{x}, \tag{106}$$

$$Y = \frac{z}{x}. \tag{107}$$

3.3.2 Deprojection

We have

$$R^2 = X^2 + Y^2 = \frac{r^2}{x^2} = \frac{1 - x^2}{x^2}, \tag{108}$$

which leads to

$$x = \frac{1}{\sqrt{1 + X^2 + Y^2}} \tag{109}$$

and using Eq. (106) and (116) we obtain

$$y = xX \tag{110}$$

$$z = xY \tag{111}$$

3.4 STG: Stereographic

The Stereographic projection is a special case of zenithal perspective projection (§3.1) with $\mu = 1$ and $\gamma = 0$.

3.4.1 Projection

We simply apply Eq. (28) and (29) with

$$R = 2\tan\frac{\rho}{2}. \tag{112}$$

We project the full sphere on the full XY-plane with a divergent point at $(\alpha, \delta) = (180, 0)$, i.e. for $x = -1$.

Using Eq. (12) we note

$$R = 2\frac{\sin\rho}{1 + \cos\rho}, \tag{113}$$

$$= 2\frac{\sin\rho}{1 + x}, \tag{114}$$

which leads to the projection formulae:

$$X = \frac{2y}{1 + x}; \tag{115}$$

$$Y = \frac{2z}{1 + x}. \tag{116}$$

3.4.2 Deprojection

From the two equiations

$$
\begin{aligned}
R^2 &= X^2 + Y^2 = \frac{4(y^2 + z^2)}{(1+x)^2}, && (117)\\
1 &= x^2 + y^2 + z^2, && (118)
\end{aligned}
$$

we deduce

$$R^2 = \frac{4(1-x^2)}{(1+x)^2} = \frac{4(1-x)}{(1+x)}, \tag{119}$$

That we solve to obtain x:

$$x = \frac{4-R^2}{4+R^2} \tag{120}$$

From the projection formulae Eq. (115) and (116) we deduce

$$
\begin{aligned}
y &= \frac{1}{2}X(1+x) = \frac{4X}{4+R^2} && (121)\\
z &= \frac{1}{2}Y(1+x) = \frac{4Y}{4+R^2} && (122)
\end{aligned}
$$

We finally find the deprojection formulae:

$$
\begin{aligned}
R' &= \frac{X^2+Y^2}{4}, && (123)\\
w &= \frac{1}{1+R'}, && (124)\\
x &= w(1-R'), && (125)\\
y &= wX, && (126)\\
z &= wY. && (127)
\end{aligned}
$$

3.5 SIN: Slant orthographic

It is the slant zenithal perspective (SZP) projection with the distance between the centre of the sphere and the projection point r_p equal to $+\infty$.

3.5.1 Projection

We use the limit of the projection equations of SZP, that is Eq. (96) and (97), when $r_p \to +\infty$:

$$
\begin{aligned}
X &= \frac{-yx_p - y_p(1-x)}{-x_p} = y + \xi_p(1-x) && (128)\\
Y &= z + \eta_p(1-x) && (129)
\end{aligned}
$$

with

$$\xi_p = \frac{y_p}{x_p} = \tan\rho_p \cos\theta_p \tag{130}$$
$$\eta_p = \frac{z_p}{x_p} = \tan\rho_p \sin\theta_p \tag{131}$$

The unitless parameters ξ and η are provided by the FITS keywords PVi_1a and PVi_2a respectively. Both their default values equal zero. But in the FIST definition,

$$\xi = \frac{\cos\theta_c}{\sin\theta_c}\sin\phi_c \tag{132}$$
$$\eta = -\frac{\cos\theta_c}{\sin\theta_c}\cos\phi_c \tag{133}$$

with the relation between (ρ_p, θ_p) and (θ_c, ϕ_c) given in §3.2, leading to (case $\mu > 0$)

$$\xi_p = -\xi \tag{134}$$
$$\eta_p = \eta \tag{135}$$

Demonstration:

$$\begin{aligned}
\xi_p &= \tan\rho_p \cos\theta_p && (136)\\
&= \frac{\sin(\frac{\pi}{2}+\theta_c)}{\cos(\frac{\pi}{2}+\theta_c)}\cos(\frac{\pi}{2}-\phi_c) && (137)\\
&= \frac{\sin(\frac{\pi}{2}-(-\theta_c))}{\cos(\frac{\pi}{2}-(-\theta_c))}\cos(\frac{\pi}{2}-\phi_c) && (138)\\
&= \frac{\cos(-\theta_c)}{\sin(-\theta_c)}\sin(\phi_c) && (139)\\
&= \frac{\cos(\theta_c)}{-\sin(\theta_c)}\sin(\phi_c) && (140)\\
&= -\xi && (141)\\
\eta_p &= \tan\rho_p \sin\theta_p && (142)\\
&= \frac{\sin(\frac{\pi}{2}+\theta_c)}{\cos(\frac{\pi}{2}+\theta_c)}\sin(\frac{\pi}{2}-\phi_c) && (143)\\
&= \frac{\cos(\theta_c)}{-\sin(\theta_c)}\cos(\phi_c) && (144)\\
&= \eta && (145)
\end{aligned}$$

From (ξ_p, η_p) we can deduce:

$$\begin{aligned}
\tan \rho_p &= -\sqrt{\xi_p^2 + \eta_p^2} && (146)\\
\rho_p &= \pi + \arctan(-\sqrt{\xi_p^2 + \eta_p^2}) = \pi - \arctan(\sqrt{\xi_p^2 + \eta_p^2}) && (147)\\
\theta_p &= \arctan 2(\frac{\eta_p}{-\sqrt{\xi_p^2 + \eta_p^2}}, \frac{\xi_p}{-\sqrt{\xi_p^2 + \eta_p^2}}) && (148)\\
\cos \theta_p &= -\frac{\xi_p}{\sqrt{\xi_p^2 + \eta_p^2}} && (149)\\
\sin \theta_p &= -\frac{\eta_p}{\sqrt{\xi_p^2 + \eta_p^2}} && (150)
\end{aligned}$$

We know that $\rho_p \in [\pi/2, \pi]$, so $\tan \rho_p$ is always negative. So the right solution of $\tan^2 \rho_p = \xi_p^2 + \eta_p^2$ is the value $-\sqrt{\xi_p^2 + \eta_p^2}$. The arctan function returns value $\in]-\pi/2, \pi/2[$ and is periodic of period π, thus $\rho_p = \pi + \arctan(-\sqrt{\xi_p^2 + \eta_p^2})$. We can thus deduce the coordiante of P on the unit sphere:

$$\begin{aligned}
x_p &= \cos \rho_p && (151)\\
&= \cos(\pi - \arctan(\sqrt{\xi_p^2 + \eta_p^2})) && (152)\\
&= -\cos \arctan(\sqrt{\xi_p^2 + \eta_p^2}) && (153)\\
&= \frac{-1}{\sqrt{1 + \xi_p^2 + \eta_p^2}} && (154)\\
y_p &= \sin \rho_p \cos \theta_p && (155)\\
&= \sin(\pi - \arctan(\sqrt{\xi_p^2 + \eta_p^2}))\frac{-\xi_p}{\sqrt{\xi_p^2 + \eta_p^2}} && (156)\\
&= \sin(\arctan(\sqrt{\xi_p^2 + \eta_p^2}))\frac{-\xi_p}{\sqrt{\xi_p^2 + \eta_p^2}} && (157)\\
&= \frac{\sqrt{\xi_p^2 + \eta_p^2}}{\sqrt{1 + \xi_p^2 + \eta_p^2}} \frac{-\xi_p}{\sqrt{\xi_p^2 + \eta_p^2}} && (158)\\
&= \frac{-\xi_p}{\sqrt{1 + \xi_p^2 + \eta_p^2}} && (159)\\
z_p &= \sin \rho_p \sin \theta_p && (160)\\
&= \frac{\sqrt{\xi_p^2 + \eta_p^2}}{\sqrt{1 + \xi_p^2 + \eta_p^2}} \frac{-\eta_p}{\sqrt{\xi_p^2 + \eta_p^2}} && (161)\\
&= \frac{-\eta_p}{\sqrt{1 + \xi_p^2 + \eta_p^2}} && (162)
\end{aligned}$$

To find this result, we use Eq. (15) and (18). We verify that $x_p^2 + y_p^2 + z_p^2 = 1$. The projection "rays" is a cylinder of axis (x_p, y_p, z_p) and radius equal to one. The projected poitn of the sphere are the points which are the nearest from the projection plane. We thus dedeuce that we can project a point only if it scalar product with (x_p, y_p, z_p) is negative

$$xx_p + yy_p + zz_p \le 0, \tag{163}$$

the point of the sphere such as $xx_p + yy_p + zz_p = 0$ beigin at the edge. They form an ellipse on the projection plane (intersection between a plane and a cylinder).

3.5.2 Deprojection

We rewrite Eq. (128) and (129) to obtain an expression of y and z:

$$\begin{aligned} y &= X - \xi_p(1-x) && (164) \\ z &= Y - \eta_p(1-x) && (165) \end{aligned}$$

We sum their square (remembering that $y^2 + z^2 = 1 - x^2$ to obtain a quadratic equation in x:

$$\begin{aligned} y^2 + z^2 &= X^2 + Y^2 + (\xi_p^2 + \eta_p^2)(1-x)^2 - 2(1-x)(\xi X + \eta Y) && (166) \\ 1 - x^2 &= x^2(\xi_p^2 + \eta_p^2) \\ &\quad +2x[(\xi X + \eta Y) - (\xi_p^2 + \eta_p^2)] \\ &\quad +(X^2 + Y^2) + (\xi_p^2 + \eta_p^2) - 2(\xi X + \eta Y) && (167) \\ 0 &= x^2(1 + \xi_p^2 + \eta_p^2) \\ &\quad +2x[(\xi X + \eta Y) - (\xi_p^2 + \eta_p^2)] \\ &\quad +(X^2 + Y^2) + (\xi_p^2 + \eta_p^2) - 2(\xi X + \eta Y) - 1 && (168) \end{aligned}$$

Thus

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{169}$$

since we keep the value having the largest x, so (nearest from the projection plane). In the previous equation the constants are:

$$\begin{aligned} \tan^2 \rho_p &= \xi_p^2 + \eta_p^2 && (170) \\ R^2 &= X^2 + Y^2 && (171) \\ R' &= \xi X + \eta Y && (172) \\ a &= (1 + \tan^2 \rho_p); && (173) \\ b &= 2(R' - \tan^2 \rho_p); && (174) \\ c &= R^2 - 2R' + \tan^2 \rho_p - 1. && (175) \end{aligned}$$

We then deduce y anf z from Eq. (164) and (165).

Testing the validity of a projection point to be de-projected: We call M a point on the projection plane of coordinates $(1, X, Y)$, O the center of the unit sphere, P the point on the unit sphere of coordinates (x_p, y_p, z_p) and A the projection of M on the plane perpendicular to $\vec{OP}$ passing through O. We have:

$$\vec{OM} = (\vec{OM}.\vec{OP})\vec{OP} + \vec{OA} \tag{176}$$

Thus

$$\vec{OA} = \begin{pmatrix} 1 \\ X \\ Y \end{pmatrix} - (x_p + y_p X + z_p Y) \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} \tag{177}$$

The point is in the projection area if $||\vec{OA}|| \leq 1$, so if

$$(1 - x_p s)^2 + (X - y_p s)^2 + (Y - z_p s)^2 \leq 1 \tag{178}$$

noting $s = x_p + y_p X + z_p Y$.

3.6 NCP: non-slant SIN with projection point located at the North Celestial Pole

Provided we set the projection center to $(0, \pi/2)$, the equation are then the same as the simple (non-slant) SIN projection, i.e.:

$$R = \sin \rho. \tag{179}$$

3.6.1 Projection

We simply have

$$X = y \tag{180}$$
$$Y = z \tag{181}$$

3.6.2 Deprojection

$$x = \sqrt{1 - (X^2 + Y^2)} \tag{182}$$
$$y = X \tag{183}$$
$$z = Y \tag{184}$$

3.7 ARC: Zenithal equidistant

The Zenithal equidistant projection is the simple case in which:

$$R = \rho \tag{185}$$

3.7.1 Projection

From Eq. (28), (29), (242) and (243), we simply have:

$$X = \rho\cos\theta = \frac{\rho}{\sin\rho}y = \frac{\arcsin\sqrt{y^2+z^2}}{\sqrt{y^2+z^2}}y = y\,\mathrm{arcsinc}\,r \tag{186}$$

$$Y = \rho\sin\theta = \frac{\rho}{\sin\rho}z = \frac{\arcsin\sqrt{y^2+z^2}}{\sqrt{y^2+z^2}}z = z\,\mathrm{arcsinc}\,r \tag{187}$$

In practice for large angular distances (let us say $\rho > \pi/2$), we replace arcsinc r by $\arccos(x)/r$ for a better numercial precision.

3.7.2 Deprojection

We easily inverse the previous equations:

$$X^2 + Y^2 = \rho^2. \tag{188}$$

Using Eq. (242), we deduce x

$$x = \cos\rho = \cos(\sqrt{X^2+Y^2}) \tag{189}$$

and from Eq. (243), (244), (28) and (29) we deduce

$$y = X\frac{\sin\rho}{\rho} = X\,\mathrm{sinc}(\sqrt{X^2+Y^2}) \tag{190}$$

$$z = Y\frac{\sin\rho}{\rho} = Y\,\mathrm{sinc}(\sqrt{X^2+Y^2}) \tag{191}$$

3.8 ZPN: Zenithal polynomial

The zenithal polynomial projection is a generalization of the zenithal equidistant projection in which the Eculidean distance is a polynomial of the angular distance

$$R = P(\rho) = p_0 + \rho(p_1 + \rho(p_2 ... + \rho(p_{n-1} + \rho p_n)...)) \tag{192}$$

3.8.1 Projection

$$r = \sin\rho = \sqrt{y^2+z^2} \tag{193}$$

$$X = P(\rho)\cos\theta = P(\rho)\frac{y}{\sin\rho} = P(\arcsin r)\frac{y}{r} \tag{194}$$

$$Y = P(\rho)\sin\theta = P(\rho)\frac{z}{\sin\rho} = P(\arcsin r)\frac{z}{r} \tag{195}$$

In practice for large angular distances (let us say $\rho > \pi/2$), we replace arcsin r by $\arccos(x)$ for a better numercial precision.

3.8.2 Deprojection

We use the Newton-Raphson's method to invert numerically Eq. (192) and find ρ from $R = \sqrt{X^2 + Y^2}$:

$$\rho = P^{-1}(R = \sqrt{X^2 + Y^2}). \tag{196}$$

To do so we first have to compute the derivative of $P'(\rho)$ of $P(\rho)$

$$R' = P'(\rho) = p_1 + \rho(2p_2 + \rho(3p_3... + \rho((n-1)p_{n-1} + \rho n p_n)...)), \tag{197}$$

then we solve iteratively $P(\rho) - R = 0$ using the recursion formula

$$\rho_{i+1} = \rho_i - \frac{P(\rho_i) - \sqrt{X^2 + Y^2}}{P'(\rho_i)} \tag{198}$$

We deduce x:

$$x = \cos\rho \tag{199}$$

and inverting the projection formulae, we find

$$y = \sqrt{\frac{1 - x^2}{X^2 + Y^2}} X \tag{200}$$

$$z = \sqrt{\frac{1 - x^2}{X^2 + Y^2}} Y \tag{201}$$

3.9 ZEA: Zenithal equal-area

The zenithal equal-area projection is the quite simple case in which

$$R = 2\sin\frac{\rho}{2}. \tag{202}$$

3.9.1 Projection

We develop the expression of the Euclidean distance R using Eq. (7) and then Eq. (242):

$$\begin{aligned} R^2 &= 4\sin^2\frac{\rho}{2} && (203) \\ &= 4\frac{1-\cos\rho}{2} && (204) \\ &= 2(1-x) && (205) \end{aligned}$$

So the projection formulae are

$$X = \sqrt{2(1-x)}\frac{y}{\sin\rho} \tag{206}$$
$$= \sqrt{\frac{2(1-x)}{1-x^2}}y \tag{207}$$
$$= \sqrt{\frac{2}{1+x}}y \tag{208}$$
$$Y = \sqrt{2(1-x)}\frac{z}{\sin\rho} \tag{209}$$
$$= \sqrt{\frac{2}{1+x}}z \tag{210}$$

3.9.2 Deprojection

We find the expression of x from $R^2 = X^2 + Y^2$:

$$R^2 = \frac{2}{1+x}r \tag{211}$$
$$= \frac{2}{1+x}(1-x^2) \tag{212}$$
$$= \frac{2}{1+x}(1+x)(1-x) \tag{213}$$
$$= 2(1-x) \tag{214}$$

So

$$x = 1 - \frac{R^2}{2} = 1 - \frac{X^2+Y^2}{2} \tag{215}$$

We now reinject this expression of x in the projection formulae Eq. (208) and (210) To obtain

$$y = \sqrt{1-\frac{R^2}{4}}X = \sqrt{1-\frac{X^2+Y^2}{4}}X \tag{216}$$
$$z = \sqrt{1-\frac{R^2}{4}}Y = \sqrt{1-\frac{X^2+Y^2}{4}}Y \tag{217}$$

3.10 AIR: Airy projection

In the Airy projection (c.f. book “Flattening the Earth: Two Thousand Years of Map Projections” by John P. Snyder):

$$R = 2\left(\frac{\ln\frac{1}{\cos\frac{\rho}{2}}}{\tan\frac{\rho}{2}} + \tan\frac{\rho}{2}\frac{\ln\frac{1}{\cos\frac{\rho_b}{2}}}{\tan^2\frac{\rho_b}{2}}\right) \tag{218}$$
$$= -2\left(\frac{\ln\cos\frac{\rho}{2}}{\tan\frac{\rho}{2}} + \tan\frac{\rho}{2}\frac{\ln\cos\frac{\rho_b}{2}}{\tan^2\frac{\rho_b}{2}}\right) \tag{219}$$

3.10.1 Projection

We use the fact that:

$$\cos\frac{\rho}{2} = \sqrt{\frac{1+\cos\rho}{2}} \tag{220}$$

$$\sin\frac{\rho}{2} = \sqrt{\frac{1-\cos\rho}{2}} \tag{221}$$

$$\tan\frac{\rho}{2} = \sqrt{\frac{1-\cos\rho}{1+\cos\rho}} \tag{222}$$

and that $x = \cos\rho$ to write

$$R = -\sqrt{\frac{1+x}{1-x}}\ln(\frac{x+1}{2}) - \sqrt{\frac{1-x}{1+x}}\frac{1+x_b}{1-x_b}\ln(\frac{x_b+1}{2}) \tag{223}$$

Thus the projection formulae are:

$$X = R\frac{y}{\sqrt{y^2+z^2}} = R\frac{y}{\sqrt{(1-x)(1+x)}} \tag{224}$$

$$= -y\left(\frac{\ln(\frac{x+1}{2})}{1-x} + \frac{1}{1+x}\frac{1+x_b}{1-x_b}\ln(\frac{x_b+1}{2})\right) \tag{225}$$

$$Y = -z\left(\frac{\ln(\frac{x+1}{2})}{1-x} + \frac{1}{1+x}\frac{1+x_b}{1-x_b}\ln(\frac{x_b+1}{2})\right) \tag{226}$$

3.10.2 Deprojection

$$\sqrt{X^2+Y^2} = \sqrt{(1-x)(1+x)}\left(\frac{\ln(\frac{x+1}{2})}{1-x} + \frac{1}{1+x}\frac{1+x_b}{1-x_b}\ln(\frac{x_b+1}{2})\right) \tag{227}$$

$$= R \tag{228}$$

$$= \sqrt{\frac{1+x}{1-x}}\ln(\frac{x+1}{2}) + \sqrt{\frac{1-x}{1+x}}\frac{1+x_b}{1-x_b}\ln(\frac{x_b+1}{2}) \tag{229}$$

We use the Newton-Raphson's algorithm to solve the equation

$$\sqrt{\frac{1+x}{1-x}}\ln(\frac{x+1}{2}) + \sqrt{\frac{1-x}{1+x}}\frac{1+x_b}{1-x_b}\ln(\frac{x_b+1}{2}) - \sqrt{X^2+Y^2} = 0 \tag{230}$$

The derivatived of the parts of the function $R(x)$ are:

$$\frac{\partial}{\partial x}\sqrt{\frac{1+x}{1-x}} = \frac{1}{(1-x)^2}\sqrt{\frac{1-x}{1+x}} \tag{231}$$

$$\frac{\partial}{\partial x}\sqrt{\frac{1-x}{1+x}} = -\frac{1}{(1+x)^2}\sqrt{\frac{1+x}{1-x}} \tag{232}$$

$$\frac{\partial}{\partial x}\ln(\frac{x+1}{2}) = \frac{1}{1+x} \tag{233}$$

We deduce the value of the derivative of $R(x)$:

$$\frac{\partial}{\partial x}R(x) = \frac{1}{1+x}\sqrt{\frac{1+x}{1-x}} \tag{234}$$

$$+\frac{1}{(1-x)^2}\sqrt{\frac{1-x}{1+x}}\ln(\frac{x+1}{2}) \tag{235}$$

$$-\frac{1}{(1+x)^2}\sqrt{\frac{1+x}{1-x}}\frac{1+x_b}{1-x_b}\ln(\frac{x_b+1}{2}) \tag{236}$$

$$R'(x) = \frac{1}{\sqrt{1-x^2}}\left(1+\frac{\ln(\frac{x+1}{2})}{1-x}-\frac{1}{1+x}\frac{1+x_b}{1-x_b}\ln(\frac{x_b+1}{2})\right) \tag{237}$$

So we find x iteratively:

$$x_{i+1} = x_i - \frac{R(x) - \sqrt{X^2+Y^2}}{R'(x)} \tag{238}$$

We deduce the two other coordinates:

$$y = \frac{X}{\sqrt{X^2+Y^2}}\sqrt{1-x^2} \tag{239}$$

$$z = \frac{Y}{\sqrt{X^2+Y^2}}\sqrt{1-x^2} \tag{240}$$

To start the iteration process, we have to start with a first approximation. We can make a very raw approximation replacing the full expression of R by:

$$R = \ln(\frac{1}{2})(1-x^2) + (1-x)\frac{1+x_b}{1-x_b}\ln(\frac{x_b+1}{2}) \tag{241}$$

This is an equation of the form $ax^2 + bx + c = 0$, ... we keep value of $x \in]-1, 1]$.

4 Cylindrical

Input:

$$x = \cos(\rho) \tag{242}$$

$$y = \sin(\rho)\cos(\theta) \tag{243}$$

$$z = \sin(\rho)\sin(\theta) \tag{244}$$

in which ρ is the angular distance between the reference point $(1, 0, 0)$ and the point to be projected (x, y, z) and θ is the angle between $\vec{y} = (0, \vec{1}, 0)$ and $(0, \vec{y}, z)$, i.e. $\theta = \frac{\pi}{2} - PA$, PA being the position angle (east of north) between the reference point and the point to be projected.

Output:

$$X = F_X(\alpha) \tag{245}$$

$$Y = F_Y(\delta) \tag{246}$$

4.1 CAR: Platte carrée

This is certainly the simplest cylindrical projection.

4.1.1 Projection

The projection formulae are

$$X = \alpha, \tag{247}$$
$$Y = \delta. \tag{248}$$

So in our framework,

$$X = \arctan 2(y, x), \tag{249}$$
$$Y = \arcsin z = \arctan 2(z, \sqrt{(x^2 + y^2)}). \tag{250}$$

For precision reasons, it is better not to use $\arcsin z$.

4.1.2 Deprojection

The de-projection formulae are

$$\alpha = X, \tag{251}$$
$$\delta = Y. \tag{252}$$

So in our framework,

$$x = \cos Y \cos X, \tag{253}$$
$$y = \cos Y \sin X, \tag{254}$$
$$z = \sin Y. \tag{255}$$

4.2 CEA: cylindrical equal area

The cylindrical equal area projection depdends on a parameter λ so it is conformal at latitudes $\delta_c = \pm \arccos \sqrt{\lambda}$. The projection formulae are

$$X = \alpha, \tag{256}$$
$$Y = \frac{\sin \delta}{\lambda}. \tag{257}$$

WARNING: X and Y must be multiplied by $180/\pi$ to be consistent with FITS WCS conventions.

4.2.1 Projection

Given the projection formula depending on (α, δ), we deduce

$$X = \arctan 2(y, x), \tag{258}$$
$$Y = \frac{z}{\lambda} \tag{259}$$

4.2.2 Deprojection

$$x = \sqrt{1 - z^2} \cos X \tag{260}$$
$$y = \sqrt{1 - z^2} \sin Y \tag{261}$$
$$z = \lambda Y \tag{262}$$

4.3 CYP: Cylindrical perspective

The cylindrical perpective projection depends on two parameters μ and λ.

- μ is the distance from the centre of the unit sphere to the projection point for a given α;
- λ is the radius of the cylinder the unit sphere is projected on.

In other words, for a position (α, δ) to be projected, the projection point as the coordinate $(\alpha, 0)$ at a distance μ of the centre of the unit sphere.

The projection formulae are

$$X = \lambda\alpha \tag{263}$$
$$Y = \frac{\mu + \lambda}{\mu + \cos\delta} \sin\delta \tag{264}$$

4.3.1 Projection

$$X = \lambda \arctan 2(y, x) \tag{265}$$
$$Y = \frac{\mu + \lambda}{\mu + \sqrt{1 - z^2}} z \tag{266}$$

4.3.2 Deprojection

5 Pseudocylindrical

5.1 MOL: Mollweide's

The Mollweide's projection is defined such that the full sky is projected inside an ellipse of surface area equal to 4π (the surface area of the unit sphere), with the half width (W, semi-major axis) equal twice the half height size (H, semi-minor axis). And the x-axis correspond to the longitude $\alpha \in [0, 2\pi[$ with $\alpha = 0$ corresponding to $X = 0$, $\alpha = \pi$ to $X = W$, $\alpha = \pi + \varepsilon$ to $X = -W + \eta$ and $\alpha = 2\pi - \varepsilon$ to $X = -\eta$. The base equations of are

$$X = W \frac{\alpha}{\pi} \cos\gamma$$
$$Y = H \sin\gamma$$

with the semi-major and semi-minor axis a and b respectively, such that

$$\begin{aligned} a &= W = 2H \\ b &= H \end{aligned}$$

and the surface area of the ellipse $\pi ab = 4\pi$ leading to $H = \sqrt{2}$ and thus to:

$$X = 2\sqrt{2}\frac{\alpha}{\pi}\cos\gamma \quad (267)$$

$$Y = \sqrt{2}\sin\gamma \quad (268)$$

This transformation is made to be equiareal. So let's compute the determinent of its Jacobian:

$$\begin{aligned} |\det J| &= \begin{vmatrix} \frac{\partial X}{\partial \alpha} & \frac{\partial X}{\partial \gamma} \\ \frac{\partial Y}{\partial \alpha} & \frac{\partial Y}{\partial \gamma} \end{vmatrix} \\ &= \begin{vmatrix} \frac{2\sqrt{2}}{\pi}\cos\gamma & -2\sqrt{2}\frac{\alpha}{\pi}\sin\gamma \\ 0 & \sqrt{2}\cos\gamma \end{vmatrix} \\ &= \frac{4}{\pi}\cos^2\gamma \end{aligned}$$

Thus

$$\mathrm{d}X\mathrm{d}Y = \frac{4}{\pi}\cos^2\gamma\mathrm{d}\alpha\mathrm{d}\gamma$$

To be equiareal, we must have (see Eq. 24):

$$\frac{4}{\pi}\cos^2\gamma\mathrm{d}\alpha\mathrm{d}\gamma \propto \cos\delta\mathrm{d}\alpha\mathrm{d}\delta$$

But we know that the surface area of the projection in the plane is equal to 4π, i.e. the same value as for orginial coordinates. So we must replace the previous proportionality by a strict equality, and we solve integrating on both sides:

$$\begin{aligned} \frac{4}{\pi}\cos^2\gamma\mathrm{d}\alpha\mathrm{d}\gamma &= \cos\delta\mathrm{d}\alpha\mathrm{d}\delta \\ \frac{4}{\pi}\int\frac{\cos(2\gamma)+1}{2}\mathrm{d}\gamma &= \int\cos\delta\mathrm{d}\delta \\ \frac{2}{\pi}(\frac{1}{2}\sin(2\gamma)+\gamma) &= \sin\delta \end{aligned}$$

So we finally find the transcendental equation

$$\pi\sin\delta = 2\gamma + \sin(2\gamma) \quad (269)$$

5.1.1 Projection

Given the previous section, to compute Mollweide's projection given a couple (α, δ), first solve the previous transcendental equation Eq. (269) using e.g. the Newton-Raphson method (numerical method). So use:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{270}$$

in which

$$\begin{aligned} f(x) &= 2x + \sin(2x) - \pi \sin\delta \\ f'(x) &= 2(1 + \cos(2x)) \end{aligned}$$

For simplicity, we may use $x' = 2x$ and (multiplying both side by a 2) use the iteration:

$$x'_{i+1} = x'_i - \frac{x'_i + \sin x'_i - \pi \sin\delta}{1 + \cos(x'_i)} \tag{271}$$

Once γ has been obtained numerically, we are able to apply the projection equations:

$$X = 2\sqrt{2}\frac{\alpha}{\pi}\cos\gamma \tag{272}$$

$$Y = \sqrt{2}\sin\gamma \tag{273}$$

5.1.2 Deprojection

We multiplying both projection equations Eq. (272) and (273), leading to

$$XY = 2\cos\gamma\sin\gamma\frac{2\alpha}{\pi}, \tag{274}$$

and using Eq. (8), we find

$$\sin(2\gamma) = XY\frac{\pi}{2\alpha}. \tag{275}$$

From Eq. (1) and then using Eq. (273) we find

$$\cos\gamma = \sqrt{1 - \sin^2\gamma} = \sqrt{1 - \frac{Y^2}{2}}. \tag{276}$$

Injecting this result in Eq. (272), we find

$$\alpha = \frac{\pi X}{2\sqrt{2 - Y^2}}. \tag{277}$$

Before apllying this equation, we have to check the special value $Y = \pm\sqrt{2}$ (or in Software $2 - Y^2 \leq 0$), in which case

$$\alpha = 0, \tag{278}$$

$$\delta = \pm\frac{\pi}{2}, \tag{279}$$

the sign of δ being the same as the sign of Y.
From Eq. (269):

$$\delta = \arcsin \frac{2\gamma + \sin(2\gamma)}{\pi}. \tag{280}$$

We have the expression of γ in Eq. (273)

$$\gamma = \arcsin \frac{Y}{\sqrt{2}}, \tag{281}$$

and from Eq. (275) and (277) we obtain

$$\sin(2\gamma) = Y\sqrt{2 - Y^2}. \tag{282}$$

We deduce from the three above equations that

$$\delta = \arcsin \frac{2 \arcsin \frac{Y}{\sqrt{2}} + Y\sqrt{2 - Y^2}}{\pi}. \tag{283}$$

The deprojection formulae are Eq. (277) and (283).

References

Calabretta, M. R. & Greisen, E. W. 2002, A&A, 395, 1077