

A Unified Approach to Proportional Navigation

CIANN-DONG YANG

CHI-CHING YANG

National Cheng Kung University
Taiwan

In this paper, the two major classes of proportional navigation (PN), namely, true proportional navigation (TPN) and pure proportional navigation (PPN) are analyzed and solved by a unified approach. The analytical tools used in the line-of-sight (LOS) referenced systems such as TPN, realistic true proportional navigation (RTPN), generalized true proportional navigation (GTPN) and ideal proportional navigation (IPN), are extended here to handle the interceptor velocity referenced systems such as PPN and its variants. It is found that the above two branches of guidance systems belong to a more general PN scheme which defines the acceleration of the interceptor as being proportional to the LOS rate with direction normal to an arbitrarily assigned vector $\vec{L}$. For example, $\vec{L}$ of TPN is LOS, and $\vec{L}$ of PPN is the interceptor's velocity. Every PN scheme associates with a specific form of $\vec{L}$. The optimal PN (OPN) problem which concerns the determination of the optimal direction $\vec{L}$ is also addressed. Under the proposed general PN scheme, its six special cases, i.e., TPN, RTPN, GTPN, IPN, PPN, and OPN are solved in a unified way from which many new relations among them can be revealed, and their performances can be compared on a common basis.

Manuscript received January 17, 1996; revised May 3 and June 28, 1996.

IEEE Log No. T-AES/33/2/1/03161.

Authors' address: Institute of Aeronautics and Astronautics, National Cheng Kung University, Tainan, Taiwan, Republic of China.

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I. INTRODUCTION

Due to its simplicity of onboard implementation, proportional navigation (PN) has attracted a considerable amount of interest in the missile guidance literature since its inception in 1940s. In its nonlinear form, the PN schemes can be categorized into two major classes: interceptor velocity referenced class and the line-of-sight (LOS) referenced class. The latter is superior to the former in the aspect of mathematical tractability, and numerous papers have appeared in the literature dealing with the analytical study of the LOS referenced system such as the true proportional navigation (TPN) [4, 5, 9, 11], generalized true proportional navigation (GTPN) [6, 13, 17], realistic true proportional navigation (RTPN) [2, 3, 15], and ideal proportional navigation (IPN) [16]. Although less success has been achieved in solving the nonlinear equations of interceptor velocity referenced system such as pure proportional navigation (PPN) [1, 7, 8, 10] and its variants, PPN is believed to be more practically implementable with no requirement on forward acceleration/deceleration and with no constraints on the initial engagement. A thorough comparison between PPN and TPN can be found in [12]. Augmented proportional navigation (APN) [18] is a linearized version of RTPN with an extra term to account for the maneuvering target. Since APN makes use of extra information, namely, knowledge of the target maneuver, it is reasonable that this knowledge should enable the missile to maneuver in a more efficient manner with less requirement of total acceleration.

One of the challenges of investigating PN guidance laws is how to formulate a standard framework under which all PN schemes can be analyzed and solved in a unified manner. The concept of generalized guidance law proposed in [14] was an attempt for this end, but was not successful enough to include PPN as its special case. Up to now, it is still a common belief that the relatively considerable success achieved in the analysis of LOS-referenced guidance laws cannot be made use of in treating the PPN problem. However, the results of this work could somewhat change this belief. Here, the problem of generalized guidance law is revisited, and the possibility of analyzing LOS-referenced guidance laws and interceptor velocity referenced guidance laws in a unified approach is verified.

The commanded acceleration of an interceptor guided by the generalized PN scheme can be expressed by the following general form.

$$\vec{a}_I = \lambda \vec{L} \times \vec{\omega} \quad (1)$$

where $\vec{L}$ is the normal direction of the commanded acceleration; $\vec{\omega} = \dot{\theta} \vec{k}$ is the LOS angular velocity, and λ is the navigation constant. Different PN guidance laws employ different forms of $\vec{L}$. For example, $\vec{L}$

of TPN is along the LOS, while $\vec{L}$ of PPN is along the interceptor's velocity. For each PN scheme, we can assign a specific $\vec{L}$ to it, and this gives rise to the optimal PN (OPN) problem that which of $\vec{L}$ is the optimal selection. Of course, the answer depends on the cost function being chosen.

Here, trajectory and performance of the interceptor guided by the generalized PN law (1) are analyzed, and its six special cases, namely, TPN, RTPN, GTPN, IPN, PPN, and OPN are solved analytically in a unified framework under which the performances such as capture area, required time of interception, and cumulative velocity increment for six PN schemes can be compared on a common basis. Using this unified approach, new relations among the six PN schemes are revealed; for example, we find that IPN is the extreme case of PPN, and that PPN resembles the behavior of OPN, etc. This work concentrates on the establishment of a unified framework for analyzing PN, and only nonmaneuvering target is considered. To account for maneuvering targets, (1) can be augmented to the following form:

$$\vec{a}_I = \lambda \vec{L} \times \vec{\omega} + \vec{a}_T$$

where $\vec{a}_T$ is the measured target's acceleration. The unified analysis given here can be applied to the case where $\vec{a}_T$ is a given constant, and a nonlinear analysis of APN can thus be derived. Closed-form solution of nonlinear APN is possible, but the expression is rather tedious. The investigation of APN using the proposed unified approach will be addressed in another paper.

II. UNIFIED PN FORMULATION

Consider a planar relative motion described by the polar coordinates (r, θ) with the origin located at the interceptor, where r is the distance between the interceptor and the target, and θ is the aspect angle of LOS with respect to an inertial reference line. Let $(\vec{e}_r, \vec{e}_\theta)$ be the unit vectors along the coordinate axes. In terms of the polar coordinates, the $\vec{L}$ directions of the six guidance laws can be expressed, according to their definitions, as

$$1) \text{ TPN: } \vec{L} = \dot{r}_0 \vec{e}_r \quad (2a)$$

$$2) \text{ RTPN: } \vec{L} = \dot{r} \vec{e}_r \quad (2b)$$

$$3) \text{ GTPN: } \vec{L} = \dot{r}_0 \vec{e}_r + r_0 \dot{\theta}_0 \vec{e}_\theta \quad (2c)$$

$$4) \text{ IPN: } \vec{L} = \dot{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta \quad (2d)$$

$$5) \text{ PPN: } \vec{L} = -\vec{V}_I \quad (2e)$$

$$6) \text{ OPN: } \vec{L} = u_{\text{opt}}(\theta) \vec{e}_r + v_{\text{opt}}(\theta) \vec{e}_\theta \quad (2f)$$

where RTPN can be conceived of the practically implemented TPN with varying closing speed, and IPN can be considered as the practically implemented GTPN with varying closing speed and with varying tangential speed. It is shown in the next section that

each $\vec{L}$ can be expressed as a function of θ :

$$\vec{L} = u(\theta) \vec{e}_r + v(\theta) \vec{e}_\theta. \quad (3)$$

The relative motion between an interceptor guided by (1) and a nonmaneuvering target is described by the following nonlinear equations

$$\ddot{r} - r\dot{\theta}^2 = -\lambda \dot{\theta} v(\theta) \quad (4a)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \lambda \dot{\theta} u(\theta). \quad (4b)$$

Without loss of generality, we assume that $\theta_0 = 0$ and $\dot{\theta}_0 > 0$ by proper choice of reference direction. With change of independent variable from t to θ , (4) can be rewritten as

$$V'_r - V_\theta = -\lambda v(\theta), \quad V_r(0) = \dot{r}_0 \quad (5a)$$

$$V'_\theta + V_r = \lambda u(\theta), \quad V_\theta(0) = r_0 \dot{\theta}_0 \quad (5b)$$

where primes denote the differentiation with respect to θ . To have the solution independent of the physical units, it is helpful to rewrite (5) in a dimensionless form

$$\bar{V}'_r - \bar{V}_\theta = -\lambda \bar{v}(\theta), \quad \bar{V}_r(0) = -\sin \psi_0 \quad (6a)$$

$$\bar{V}'_\theta + \bar{V}_r = \lambda \bar{u}(\theta), \quad \bar{V}_\theta(0) = \cos \psi_0 \quad (6b)$$

where the variables with overline denote the division by $V_0 = \sqrt{\dot{r}_0^2 + (r_0 \dot{\theta}_0)^2}$; for instance, $\bar{V}_r = V_r/V_0$, etc. The dimensionless time τ is defined as $\tau = t/(r_0/V_0)$, and the parameter ψ_0 is defined as $\psi_0 = \arccos(r_0 \dot{\theta}_0 / \sqrt{\dot{r}_0^2 + (r_0 \dot{\theta}_0)^2})$. For given functions of $\bar{u}(\theta)$ and $\bar{v}(\theta)$ corresponding to a specific guidance law, (6) can be solved to obtain $\bar{V}_r(\theta)$ and $\bar{V}_\theta(\theta)$ which, in turn, can be used to evaluate the capture area, the cumulative velocity increment, and the required time of interception.

Once $\bar{V}_r(\theta)$ and $\bar{V}_\theta(\theta)$ are found, the capture area is determined from the conditions:

$$\bar{V}_r(\theta_f) \leq 0, \quad \bar{V}_\theta(\theta_f) = 0 \quad (7)$$

and the normalized cumulative velocity increment is calculated as

$$\begin{aligned} \Delta \bar{V} &= \int_0^{\tau_f} \left| \frac{a_I}{V_0^2/r_0} \right| d\tau \\ &= \lambda \int_0^{\theta_f} \sqrt{\bar{u}^2(\theta) + \bar{v}^2(\theta)} d\theta. \end{aligned} \quad (8)$$

To derive the trajectory of the relative motion, we introduce the unit angular momentum h :

$$h = r^2 \dot{\theta} \quad (9)$$

and the associated dimensionless form is $\bar{h} = h/(r_0 V_0)$. In terms of h , (6) is combined to yield

$$\frac{d\bar{h}}{\bar{h}} = \frac{\bar{V}'_\theta(\theta) + \bar{V}_r(\theta)}{\bar{V}_\theta(\theta)} d\theta. \quad (10a)$$

The integration gives

$$\bar{h}(\theta) = \bar{h}_0 e^{-\Phi(\theta)} \quad (10b)$$

where $\bar{h}_0 = \cos \psi_0$. The function $\Phi(\theta)$ is called the characteristic function of the guidance law, and is defined as

$$\Phi(\theta) = - \int_0^\theta \frac{\bar{V}'_\theta(\phi) + \bar{V}_r(\phi)}{\bar{V}_\theta(\phi)} d\phi. \quad (11)$$

Each guidance law possesses a unique characteristic function from which all the trajectory properties are inherited. The characteristic functions for the six guidance laws are derived in the next section. After obtaining $\bar{h}(\theta)$, we can find the LOS angular rate from the following relation:

$$\frac{d\theta}{d\tau} = \frac{\bar{V}_\theta^2}{\bar{h}}. \quad (12)$$

Integrating both sides gives the required time of interception

$$\tau_f = \int_0^{\theta_f} \frac{d\theta}{d\theta/d\tau} = \bar{h}_0 \int_0^{\theta_f} \frac{e^{-\Phi(\theta)}}{\bar{V}_\theta^2(\theta)} d\theta. \quad (13)$$

All the six guidance laws can be analyzed by the same procedures mentioned above. Under this unified approach, the only difference in analyzing these guidance laws is the vector $\bar{L}$. By considering six types of $\bar{L}$, the performances of the corresponding six guidance laws can be evaluated and compared in a standard framework.

III. PERFORMANCE EVALUATION OF SIX GUIDANCE LAWS

A. TPN

From the definition of TPN, we have $\bar{L}_{TPN} = \dot{r}_0 \bar{e}_r$, i.e., $u(\theta) = \dot{r}_0$ and $v(\theta) = 0$, or in dimensionless form: $\bar{u}(\theta) = -\sin \psi_0$, $\bar{v}(\theta) = 0$. Substituting $\bar{u}(\theta)$ and $\bar{v}(\theta)$ into (6), and solving the resulting coupled linear equations, we obtain readily

$$\bar{V}_r(\theta) = R \sin(\theta + \delta) - \lambda \sin \psi_0 \quad (14a)$$

$$\bar{V}_\theta(\theta) = R \cos(\theta + \delta) \quad (14b)$$

where

$$R = \sqrt{k^2 \sin^2 \psi_0 + \cos^2 \psi_0}, \quad \tan \delta = k \tan \psi_0.$$

The definition $k = \lambda - 1$ is used throughout this work. The condition $\bar{V}_\theta(\theta_f) = 0$ gives the terminal aspect angle θ_f as

$$\theta_f = \frac{\pi}{2} - \delta = \text{arccot}(k \tan \psi_0). \quad (15)$$

Using above θ_f in the capture condition $\bar{V}_r(\theta_f) \leq 0$ and expressing the resultant in terms of $\bar{h}_0$, we obtain

the normalized capture area for TPN:

$$0 \leq \bar{h}_0 \leq \sqrt{1 - \frac{1}{2\lambda}}. \quad (16)$$

The name ‘‘capture area’’ originated from the fact that the admissible engagement conditions are described as an area in the (V_{r_0}, V_{θ_0}) plane. However, we recognize that the name ‘‘capture length’’ is more suitable for the present approach, since the admissible engagement conditions are expressed as a length interval of $\bar{h}_0$. To be consistent with the conventional usage, we still adopt the name capture area in the following discussion, but we want to remind the readers of the fact that when capture area is normalized to dimensionless form, it becomes a length not an area. Capture length can be interpreted as a quantitative measure of capture area and makes it possible the quantitative comparison of capture areas for different PN laws. Recall that the sample space of capture length is $\{\bar{h}_0 \mid 0 \leq \bar{h}_0 \leq 1\}$, hence, the capture length of TPN with $\lambda = 2$ covers $\sqrt{3}/2$ portion of the whole sample space. That is to say, when the engagement starts with random initial condition, the probability of successful interception is about $\sqrt{3}/2$.

The cumulative velocity increment for TPN comes from (8) with $\bar{u}(\theta) = -\sin \psi_0$ and $\bar{v}(\theta) = 0$

$$\begin{aligned} \Delta \bar{V}_{TPN} &= \lambda \int_0^{\theta_f} \sqrt{\bar{u}^2(\theta) + \bar{v}^2(\theta)} d\theta \\ &= \lambda \theta_f \sin \psi_0. \end{aligned} \quad (17)$$

The characteristic function for TPN is found by substituting (14) into (11)

$$\Phi_{TPN} = \frac{\lambda \sin \psi_0}{R} \ln \frac{\tan\left(\frac{\theta + \delta}{2} + \frac{\pi}{4}\right)}{\tan\left(\frac{\delta}{2} + \frac{\pi}{4}\right)}. \quad (18)$$

Substituting Φ_{TPN} into (13), we obtain the required time of interception

$$\tau_{fTPN} = \frac{\bar{h}_0}{2R} \left[\frac{\cot(\delta/2 + \pi/4)}{\lambda \sin \psi_0 + R} + \frac{\tan(\delta/2 + \pi/4)}{\lambda \sin \psi_0 - R} \right]. \quad (19)$$

B. RTPN

RTPN is the practical implementation of TPN using instantaneous closing speed in the interceptor’s acceleration. For RTPN, we have $\bar{L}_{RTPN} = \dot{r} \bar{e}_r$, i.e., $\bar{u} = \bar{V}_r$ and $\bar{v} = 0$ which are then substituted into (6) to yield the governing equations of RTPN

$$\bar{V}'_r - \bar{V}_\theta = 0 \quad (20a)$$

$$\bar{V}'_\theta + \bar{V}_r = \lambda \bar{V}_r. \quad (20b)$$

These coupled linear differential equations can be solved easily, leading to the following solutions

$$\bar{V}_r(\theta) = \frac{\cos \psi_0}{\sqrt{k}} \sinh \sqrt{k} \theta - \sin \psi_0 \cosh \sqrt{k} \theta \quad (21a)$$

$$\bar{V}_\theta(\theta) = \cos \psi_0 \cosh \sqrt{k} \theta - \sqrt{k} \sin \psi_0 \sinh \sqrt{k} \theta. \quad (21b)$$

The terminal aspect angle θ_f evaluated from the condition $\bar{V}_\theta(\theta_f) = 0$ is found as

$$\theta_f = \frac{1}{2\sqrt{k}} \ln \frac{\sqrt{k} \sin \psi_0 + \cos \psi_0}{\sqrt{k} \sin \psi_0 - \cos \psi_0} \quad (22)$$

which can be used in the capture condition $\bar{V}_r(\theta_f) \leq 0$ to derive the capture area of RTPN

$$0 \leq \bar{h}_0 < \sqrt{1 - \frac{1}{\lambda}}. \quad (23)$$

As expected, the capture area of RTPN is smaller than that of TPN, when we compare (23) with (16). Using (8), the cumulative velocity increment of RTPN becomes

$$\Delta \bar{V}_{\text{RTPN}} = -\lambda \int_0^{\theta_f} \bar{V}_r(\theta) d\theta = \frac{\lambda}{k} \cos \psi_0. \quad (24)$$

Next we consider the required time of interception of RTPN. The characteristic function of RTPN is determined by substituting (21) into (11)

$$\Phi_{\text{RTPN}}(\theta) = -\frac{\lambda}{k} \ln |\cosh \sqrt{k} \theta - \sqrt{k} \tan \psi_0 \sinh \sqrt{k} \theta| \quad (25)$$

which, in turn, is applied in (13) to obtain $\tau_{f\text{RTPN}}$ as

$$\tau_{f\text{RTPN}} = \bar{h}_0^{-1/\lambda-1} \int_0^{\theta_f} (\cos \psi_0 \cosh \sqrt{k} \theta - \sqrt{k} \sin \psi_0 \sinh \sqrt{k} \theta)^{(2-\lambda)/\lambda-1} d\theta. \quad (26)$$

A more concise expression of $\tau_{f\text{RTPN}}$ can be found by exploiting the following relation between θ and the normalized relative distance ρ

$$(\cos \psi_0 - \sqrt{k} \sin \psi_0) e^{\sqrt{k} \theta} = \bar{h}_0 \rho^k - \sqrt{\bar{h}_0^2 \rho^{2k} - \lambda \bar{h}_0^2 + k}.$$

In terms of ρ , the integration in (26) becomes

$$\tau_{f\text{RTPN}} = \int_0^1 \frac{\sqrt{k} d\rho}{\sqrt{\bar{h}_0^2 \rho^{2k} - \lambda \bar{h}_0^2 + k}}. \quad (27)$$

C. GTPN

GTPN is another generation of TPN, where the interceptor's acceleration keeps a fixed angle with respect to the normal direction of LOS. We have from (1c) that $\bar{L}_{\text{GTPN}} = \bar{V}_0 = \dot{r}_0 \bar{e}_r + r_0 \dot{\theta}_0 \bar{e}_\theta$, i.e., $u(\theta) = \dot{r}_0$ and

$v(\theta) = r_0 \dot{\theta}_0$, or in dimensionless form: $\bar{u}(\theta) = -\sin \psi_0$, $\bar{v}(\theta) = \cos \psi_0$. Substituting $\bar{u}(\theta)$ and $\bar{v}(\theta)$ into (6) yields the governing equations of GTPN

$$\bar{V}'_r - \bar{V}_\theta = -\lambda \cos \psi_0 \quad (28a)$$

$$\bar{V}'_\theta + \bar{V}_r = -\lambda \sin \psi_0 \quad (28b)$$

whose solutions can be derived easily as

$$\bar{V}_r(\theta) = k \sin(\theta + \delta) - \lambda \sin \psi_0 \quad (29a)$$

$$\bar{V}_\theta(\theta) = k \cos(\theta + \delta) + \lambda \cos \psi_0 \quad (29b)$$

where $\delta = \pi - \psi_0$. The condition $\bar{V}_\theta(\theta_f) = 0$ gives the terminal aspect angle θ_f as

$$\theta_f = \arccos \left(\frac{-\lambda}{k} \cos \psi_0 \right) - \pi + \psi_0 \quad (30)$$

and the inequality $\bar{V}_r(\theta_f) \leq 0$ is reduced to the following form

$$0 \leq \bar{h}_0 \leq 1 - \frac{1}{\lambda} \quad (31)$$

which is the capture area of GTPN. An alternative definition of GTPN [6] is $\bar{u}(\theta) = C_1$ and $\bar{v}(\theta) = C_2$, where constants C_1 and C_2 are independent of initial conditions. Since one more free parameter is introduced in this case, the capture area can be larger than that derived above. As to the cumulative velocity increment for GTPN, we have from (8)

$$\Delta \bar{V}_{\text{GTPN}} = \lambda \int_0^{\theta_f} \sqrt{\bar{u}^2(\theta) + \bar{v}^2(\theta)} d\theta = \lambda \theta_f \quad (32)$$

where θ_f has been derived in (30). Substituting (29) into (11) yields the characteristic function of GTPN

$$\begin{aligned} \Phi_{\text{GTPN}}(\theta) &= \int_0^{\theta_f} \frac{\lambda \sin \psi_0}{k \cos(\theta + \delta) + \lambda \cos \psi_0} d\theta \\ &= F(\theta + \delta) - F(\delta) \end{aligned} \quad (33)$$

where

$$\begin{aligned} F(x) &= \frac{\lambda \sin \psi_0}{\sqrt{k^2 - \lambda^2 \cos^2 \psi_0}} \\ &\times \ln \left| \frac{(k - \lambda \cos \psi_0) \tan(x/2) + \sqrt{k^2 - \lambda^2 \cos^2 \psi_0}}{(k - \lambda \cos \psi_0) \tan(x/2) - \sqrt{k^2 - \lambda^2 \cos^2 \psi_0}} \right|. \end{aligned}$$

Having obtained $\Phi_{\text{GTPN}}(\theta)$, we proceed to find the required time of interception of GTPN by using $\Phi_{\text{GTPN}}(\theta)$ in (13)

$$\tau_{f\text{GTPN}} = \bar{h}_0 \int_0^{\theta_f} \frac{e^{-\Phi_{\text{GTPN}}(\theta)}}{(k \cos(\theta + \delta) + \lambda \cos \psi_0)^2} d\theta. \quad (34)$$

D. IPN

Just like the fact that RTPN is the implemented version of TPN, one can conceive of IPN as the implemented version of GTPN. Instead of $\bar{L}_{\text{GTPN}} = \bar{V}_0$

used in GTPN, IPN employs the instantaneous relative velocity $\vec{L}_{IPN} = \vec{V}(\theta)$ to generate the interceptor's acceleration. Hence, in the case of IPN, we have $u(\theta) = \dot{r}$ and $v(\theta) = r\dot{\theta}$, and (6) becomes

$$\vec{V}_r' - \vec{V}_\theta = -\lambda \vec{V}_\theta \quad (35a)$$

$$\vec{V}_\theta' + \vec{V}_r = \lambda \vec{V}_r. \quad (35b)$$

Solving for $\vec{V}_r(\theta)$ and $\vec{V}_\theta(\theta)$, we obtain

$$\vec{V}_r(\theta) = -\sin(k\theta + \psi_0), \quad \vec{V}_\theta(\theta) = \cos(k\theta + \psi_0). \quad (36)$$

The terminal aspect angle θ_f evaluated from the condition $\vec{V}_\theta(\theta_f) = 0$ is

$$\theta_f = \frac{1}{k} \left(\frac{\pi}{2} - \psi_0 \right) \quad (37)$$

and the capture condition $\vec{V}_r(\theta_f) = -1 \leq 0$ is always satisfied for any initial conditions of engagement, as long as $\lambda > 1$. Hence, the capture area of IPN covers all range of $\bar{h}_0$, i.e., $0 \leq \bar{h}_0 \leq 1$. Next we consider the cumulative velocity increment of IPN. From (8), we have

$$\Delta \vec{V}_{IPN} = \lambda \theta_f = \frac{\lambda}{k} \left(\frac{\pi}{2} - \psi_0 \right). \quad (38)$$

It can be seen that $\Delta \vec{V}_{IPN} \rightarrow 0$, as $\psi_0 \rightarrow \pi/2$. This corresponds to the tail-chase condition. On the other hand, $\Delta \vec{V}_{IPN}$ achieves its maximum as ψ_0 approaches 0, which corresponds to the case that the target escapes initially along the normal direction of LOS. As to the characteristic function of IPN, we substitute (36) into (11) to obtain

$$\Phi_{IPN}(\theta) = \frac{-\lambda}{k} \ln \frac{\cos(k\theta + \psi_0)}{\cos \psi_0}. \quad (39)$$

We can determine the required time of interception of IPN from the (13) with $\vec{V}_\theta(\theta)$ and $\Phi(\theta)$ given by (36) and (39), respectively.

E. PPN

The relative motion of PPN adopts the interceptor's velocity as the reference line. Therefore, it is a common belief that PPN cannot be analyzed using the LOS-based coordinates. However, we will show below that the unified methodology, which employs the LOS-based framework and is applied successfully to the above four guidance laws, is also applicable to the PPN case. The first step in this formulation is to express the interceptor's velocity $\vec{V}_I$ in terms of the LOS-based coordinates. Referring to Fig. 1, we have

$$\vec{V}_I = V_I \cos(\theta - \alpha) \vec{e}_r - V_I \sin(\theta - \alpha) \vec{e}_\theta \quad (40)$$

where α is the aspect angle of $\vec{V}_I$ with respect to the inertial reference line. From the definition of PPN, the magnitude of the interceptor's acceleration a_I is $a_I = V_I \dot{\alpha} = \lambda V_I \dot{\theta}$. Hence, α is related to θ

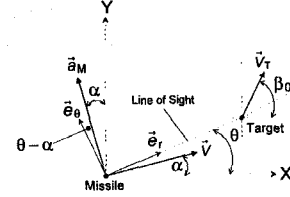


Fig. 1. Planar pursuit geometry.

via $\alpha = \lambda \theta + \alpha_0$. Now the normal direction $\vec{L}_{PPN}$ of the acceleration vector $\vec{a}_I$ can be expressed in the LOS-based coordinates as $\vec{L}_{PPN} = -\vec{V}_I = -V_I \cos(k\theta + \alpha_0) \vec{e}_r - V_I \sin(k\theta + \alpha_0) \vec{e}_\theta$, i.e.,

$$u(\theta) = -V_I \cos(k\theta + \alpha_0), \quad v(\theta) = -V_I \sin(k\theta + \alpha_0). \quad (41)$$

Substituting (41) into (6) yields the governing equations of PPN

$$\vec{V}_r' - \vec{V}_\theta = \lambda \vec{V}_I \sin(k\theta + \alpha_0), \quad \vec{V}_r(0) = -\sin \psi_0 \quad (42a)$$

$$\vec{V}_\theta' + \vec{V}_r = -\lambda \vec{V}_I \cos(k\theta + \alpha_0), \quad \vec{V}_\theta(0) = \cos \psi_0. \quad (42b)$$

These coupled linear differential equations can be solved readily for $\vec{V}_r(\theta)$ and $\vec{V}_\theta(\theta)$ as

$$\begin{aligned} \vec{V}_r(\theta) = & (\vec{V}_r(0) + \vec{V}_I \cos \alpha_0) \cos \theta \\ & + (\vec{V}_\theta(0) + \vec{V}_I \sin \alpha_0) \sin \theta - \vec{V}_I \cos(k\theta + \alpha_0) \end{aligned} \quad (43a)$$

$$\begin{aligned} \vec{V}_\theta(\theta) = & (\vec{V}_r(0) + \vec{V}_I \cos \alpha_0) \sin \theta \\ & + (\vec{V}_\theta(0) + \vec{V}_I \sin \alpha_0) \cos \theta - \vec{V}_I \sin(k\theta + \alpha_0). \end{aligned} \quad (43b)$$

Above solutions are in terms of the initial conditions $\vec{V}_r(0)$ and $\vec{V}_\theta(0)$. To compare with the conventional result where solutions are expressed in terms of the initial orientations of $\vec{V}_I$ and $\vec{V}_T$, we need to relate $\vec{V}_r(0)$ and $\vec{V}_\theta(0)$ to α_0 and β_0 . Referring to (1), we have

$$\vec{V}_r(0) = \vec{V}_T \cos \beta_0 - \vec{V}_I \cos \alpha_0 \quad (44a)$$

$$\vec{V}_\theta(0) = \vec{V}_T \sin \beta_0 - \vec{V}_I \sin \alpha_0 \quad (44b)$$

Using (44) in (43) leads to the familiar result:

$$\vec{V}_r(\theta) = \vec{V}_T \cos(\beta_0 - \theta) - \vec{V}_I \cos(\theta - \alpha) \quad (45a)$$

$$\vec{V}_\theta(\theta) = \vec{V}_T \sin(\beta_0 - \theta) + \vec{V}_I \sin(\theta - \alpha). \quad (45b)$$

Though the identical result can also be obtained from direct observation of the kinematic diagram, here we have rederived the key equations of PPN using a unified framework where LOS-based guidance laws such as TPN, RTPN, GTPN, and IPN have been analyzed.

Define two parameters ζ and η such that $\beta_0 = \zeta\alpha_0$, and $V_I = \eta V_T$, then it can be shown that $\bar{V}_T = 1/(\eta - 1)$, $\bar{V}_I = \eta/(\eta - 1)$, and $\alpha_0 = \psi_0 - \pi/2$. Accordingly, (45) can be rewritten as

$$\bar{V}_r(\theta) = -\frac{1}{\eta-1} \sin(\theta - \psi_0) - \frac{\eta}{\eta-1} \sin(k\theta + \psi_0) \quad (46a)$$

$$\bar{V}_\theta(\theta) = \frac{-1}{\eta-1} \cos(\theta - \psi_0) + \frac{\eta}{\eta-1} \cos(k\theta + \psi_0) \quad (46b)$$

where for comparison purpose, we only consider the case of $\zeta = 1$. When other values of ζ are concerned, they can be treated in the same way. The terminal aspect angle θ_f is determined from the condition $\bar{V}_\theta(\theta_f) = 0$, i.e.,

$$\cos(\theta_f - \psi_0) = \eta \cos(k\theta_f + \psi_0). \quad (47)$$

It was shown by qualitative methods [7] that $\bar{V}_r(\theta_f) \leq 0$ is always satisfied for any initial conditions, provided $\lambda > 2$ and $\eta > 1$. Hence, the capture area of PPN covers all range of $\bar{h}_0$, i.e., $0 \leq \bar{h}_0 \leq 1$. Regarding to the cumulative velocity increment of PPN, we have the following result from (8) and (41)

$$\begin{aligned} \Delta \bar{V}_{PPN} &= \lambda \int_0^{\theta_f} \sqrt{\bar{u}^2(\theta) + \bar{v}^2(\theta)} d\theta \\ &= \lambda \bar{V}_I \theta_f = \frac{\lambda \eta}{\eta - 1} \theta_f. \end{aligned} \quad (48)$$

The characteristic function of PPN can be found by substituting (46) into (11), leading to

$$\Phi_{PPN}(\theta) = \int_0^\theta \frac{\lambda \eta \sin(k\theta + \psi_0)}{\eta \cos(k\theta + \psi_0) - \cos(\theta - \psi_0)} d\theta \quad (49)$$

$\Phi_{PPN}(\theta)$ is then exploited to determine the required time of interception by using (13).

F. OPN

The interceptor's acceleration of PN guidance laws possesses the general form: $\ddot{a}_I = \lambda \bar{L} \times \bar{\omega}$, where $\bar{L} = u(\theta)\bar{e}_r + v(\theta)\bar{e}_\theta$. In the above discussion we have considered five PN guidance laws, with each guidance law associated with a specific pair of $(u(\theta), v(\theta))$. It is natural to raise the question that which pair of $(u(\theta), v(\theta))$ is the best choice? The answer depends on the performance index used for the optimization. Here we adopt the following performance index:

$$I = \theta_f + \gamma \int_0^{\theta_f} (\bar{u}^2(\theta) + \bar{v}^2(\theta)) d\theta \quad (50)$$

where the first term in I takes into account the time of interception, since θ is a monotonically increasing or decreasing function of time t as can be seen from

(12). The second term in I measures the energy consumption during the interception. The optimal $\bar{u}(\theta)$ and $\bar{v}(\theta)$ can be determined using the standard variational approach. We rewrite $\bar{u}$ and $\bar{v}$ in the way $\bar{u} = -M \cos \sigma$, $\bar{v} = -M \sin \sigma$. Hence, the governing equations (6) become

$$\bar{V}'_r - \bar{V}_\theta = -\lambda \bar{v} = \lambda M \sin \sigma \quad (51a)$$

$$\bar{V}'_\theta + \bar{V}_r = \lambda \bar{u} = -\lambda M \cos \sigma \quad (51b)$$

and the corresponding Hamiltonian is

$$\begin{aligned} H &= \gamma M^2 + \lambda_{\bar{V}_r} (\bar{V}_\theta + \lambda M \sin \sigma) \\ &\quad + \lambda_{\bar{V}_\theta} (-\bar{V}_r - \lambda M \cos \sigma) \end{aligned} \quad (52)$$

where the adjoint variables $\lambda_{\bar{V}_r}$ and $\lambda_{\bar{V}_\theta}$ satisfy

$$\lambda'_{\bar{V}_r} = -\frac{\partial H}{\partial \bar{V}_r} = \lambda_{\bar{V}_\theta}, \quad \lambda_{\bar{V}_r}(\theta_f) = 0 \quad (53a)$$

$$\lambda'_{\bar{V}_\theta} = -\frac{\partial H}{\partial \bar{V}_\theta} = -\lambda_{\bar{V}_r}, \quad \lambda_{\bar{V}_\theta}(\theta_f) = \text{free}. \quad (53b)$$

The solutions of (53) can be easily derived as

$$\begin{aligned} \lambda_{\bar{V}_r}(\theta) &= A \sin(\theta - \theta_f), \\ \lambda_{\bar{V}_\theta}(\theta) &= A \cos(\theta - \theta_f) \end{aligned} \quad (54)$$

where A and θ_f are to be determined. The conditions $\partial H / \partial M = 0$ and $\partial H / \partial \sigma = 0$ give, respectively, the optimal M and σ as

$$\begin{aligned} M^* &= \frac{\lambda}{2\gamma} (\lambda_{\bar{V}_\theta} \cos \sigma - \lambda_{\bar{V}_r} \sin \sigma), \\ \tan \sigma^* &= -\frac{\lambda_{\bar{V}_r}}{\lambda_{\bar{V}_\theta}}. \end{aligned} \quad (55)$$

The combination of (54) and (55) reveals that $M^* = \lambda A (2\gamma)^{-1}$ and $\sigma^* = \theta_f - \theta$. The optimal pair of $(\bar{u}(\theta), \bar{v}(\theta))$ turns out to be

$$\begin{aligned} \bar{u}^*(\theta) &= -\frac{\lambda A}{2\gamma} \cos(\theta - \theta_f), \\ \bar{v}^*(\theta) &= \frac{\lambda A}{2\gamma} \sin(\theta - \theta_f). \end{aligned} \quad (56)$$

The optimal trajectory is governed by (51) with $\bar{u}$ and $\bar{v}$ replaced by (56).

$$\bar{V}'_r - \bar{V}_\theta = +\frac{\lambda^2 A}{2\gamma} \sin(-\theta + \theta_f) \quad (57a)$$

$$\bar{V}'_\theta + \bar{V}_r = -\frac{\lambda^2 A}{2\gamma} \cos(-\theta + \theta_f). \quad (57b)$$

The similarity between (42) and (57) manifests the close connection between PPN and OPN. This point is investigated in the next section. Solving (57) with the initial conditions $\bar{V}_r(0) = -\sin \psi_0$ and $\bar{V}_\theta(0) = \cos \psi_0$

and with the terminal conditions $\bar{V}_\theta(\theta_f) = 0$ and $H(\theta_f) = -1$, we obtain

$$\begin{aligned}\bar{V}_r(\theta) = & +\sin(\theta_f - \psi_0)\cos(\theta - \theta_f) \\ & - \frac{\lambda^2 A}{2\gamma}(\theta - \theta_f)\sin(\theta - \theta_f)\end{aligned}\quad (58a)$$

$$\begin{aligned}\bar{V}_\theta(\theta) = & -\sin(\theta_f - \psi_0)\sin(\theta - \theta_f) \\ & - \frac{\lambda^2 A}{2\gamma}(\theta - \theta_f)\cos(\theta - \theta_f)\end{aligned}\quad (58b)$$

where $A = 2\gamma(\lambda^2\theta_f)^{-1}\cos(\theta_f - \psi_0)$ and θ_f is the smallest root satisfying

$$\frac{\lambda^2}{\gamma}\theta_f^2 = \theta_f \sin 2(\theta_f - \psi_0) + \cos^2(\theta_f - \psi_0). \quad (59)$$

The capture area described by $\bar{V}_r(\theta_f) \leq 0$ becomes

$$0 \leq \bar{h}_0 \leq \cos \theta_f. \quad (60)$$

The cumulative velocity increment of OPN is calculated from (8) and (56)

$$\begin{aligned}\Delta \bar{V}_{\text{OPN}} &= \lambda \int_0^{\theta_f} \sqrt{\bar{u}^2(\theta) + \bar{v}^2(\theta)} d\theta \\ &= \frac{\lambda^2 A}{2\gamma} \theta_f = \cos(\theta_f - \psi_0).\end{aligned}\quad (61)$$

The substitution of (58) into (11) yields the characteristic function of OPN as

$$\Phi_{\text{OPN}}(\theta) = \int_0^{\theta_f} \frac{\cos(\theta_f - \psi_0)\cos(\theta - \theta_f)}{\theta_f \cos(\theta - \psi_0) - \theta \cos(\theta_f - \psi_0)\cos(\theta - \theta_f)} d\theta. \quad (62)$$

IV. COMPARISON AND RELATIONSHIP

In the previous section, the performance and trajectories of the six guidance laws have been analyzed in a unified framework. For the sake of comparison and for the convenience of deducing the relationship among the guidance laws, it is better to put together the characteristics of the various guidance laws.

A) *Acceleration Normal Direction*: $\vec{L} = \bar{u}(\theta)\vec{e}_r + \bar{v}(\theta)\vec{e}_\theta$.

A1) TPN:

$$\bar{u}(\theta) = -\sin \psi_0, \quad \bar{v}(\theta) = 0$$

A2) RTPN:

$$\bar{u}(\theta) = \frac{\cos \psi_0}{\sqrt{k}} \sinh \sqrt{k}\theta - \sin \psi_0 \cosh \sqrt{k}\theta, \quad \bar{v}(\theta) = 0$$

A3) GTPN:

$$\bar{u}(\theta) = -\sin \psi_0, \quad \bar{v}(\theta) = \cos \psi_0$$

A4) IPN:

$$\bar{u}(\theta) = -\sin(k\theta + \psi_0), \quad \bar{v}(\theta) = \cos(k\theta + \psi_0)$$

A5) PPN:

$$\bar{u}(\theta) = -\frac{\eta}{\eta - 1} \sin(k\theta + \psi_0),$$

$$\bar{v}(\theta) = \frac{\eta}{\eta - 1} \cos(k\theta + \psi_0)$$

A6) OPN:

$$\bar{u}(\theta) = \frac{-1}{\lambda\theta_f} \cos(\theta_f - \psi_0) \cos(\theta - \theta_f),$$

$$\bar{v}(\theta) = \frac{1}{\lambda\theta_f} \cos(\theta_f - \psi_0) \sin(\theta - \theta_f)$$

Three remarks can be addressed from the trajectories of $\vec{L}(\theta)$ within the inertial $X - Y$ coordinates for various guidance laws. 1) The trajectories of $\vec{L}_{\text{PPN}}$ with varying speed ratio η converge to the trajectory of $\vec{L}_{\text{IPN}}$, as η approaches infinity. This observation reflects the fact that IPN is the extreme case of PPN. 2) The trajectory of $\vec{L}_{\text{GTPN}}$ coincides with the trajectory of $\vec{L}_{\text{IPN}}$. In fact, both trajectories lie on the unit circle, because of $\bar{u}^2(\theta) + \bar{v}^2(\theta) = 1$ for both case. However, it should be noted that at any instance the orientations of $\vec{L}_{\text{GTPN}}$ and $\vec{L}_{\text{IPN}}$ may not coincide. 3) Of most significance is that the orientation of $\vec{L}_{\text{OPN}}$ is fixed in the inertial coordinates. This observation can be checked theoretically by showing $\vec{L}_{\text{OPN}}$ is a constant vector.

B) *Terminal Aspect Angle*: θ_f .

B1) TPN:

$$\theta_f = \text{arccot}(k \tan \psi_0)$$

B2) RTPN:

$$\theta_f = \frac{1}{2\sqrt{k}} \ln \frac{\sqrt{k} \sin \psi_0 + \cos \psi_0}{\sqrt{k} \sin \psi_0 - \cos \psi_0}$$

B3) GTPN:

$$\theta_f = \arccos\left(\frac{-\lambda}{k} \cos \psi_0\right) + \psi_0 - \pi$$

B4) IPN:

$$\theta_f = \frac{1}{k} \left(\frac{\pi}{2} - \psi_0\right)$$

B5) PPN:

$$\cos(\theta_f - \psi_0) = \eta \cos(k\theta_f + \psi_0)$$

B6) OPN:

$$\frac{\lambda^2}{\gamma}\theta_f^2 = \theta_f \sin 2(\theta_f - \psi_0) + \cos^2(\theta_f - \psi_0)$$

C) *Cumulative Velocity Increment*: $\Delta \bar{V}$.

C1) TPN:

$$\Delta \bar{V} = \lambda \theta_f \sin \psi_0$$

C2) RTPN:

$$\Delta \bar{V} = \frac{\lambda}{k} \cos \psi_0$$

C3) GTPN:

$$\Delta \bar{V} = \lambda \theta_f$$

C4) IPN:

$$\Delta \bar{V} = \frac{\lambda}{k} \left(\frac{\pi}{2} - \psi_0 \right)$$

C5) PPN:

$$\Delta \bar{V} = \frac{\lambda \eta}{\eta - 1} \theta_f$$

C6) OPN:

$$\Delta \bar{V} = \cos(\theta_f - \psi_0)$$

D) Capture Area.

D1) TPN:

$$\left\{ \bar{h}_0 \mid 0 \leq \bar{h}_0 \leq \sqrt{1 - \frac{1}{2\lambda}} \right\}$$

D2) RTPN:

$$\left\{ \bar{h}_0 \mid 0 \leq \bar{h}_0 \leq \sqrt{1 - \frac{1}{\lambda}} \right\}$$

D3) GTPN:

$$\left\{ \bar{h}_0 \mid 0 \leq \bar{h}_0 \leq 1 - \frac{1}{\lambda} \right\}$$

D4) IPN:

$$\left\{ \bar{h}_0 \mid 0 \leq \bar{h}_0 \leq 1 \right\}$$

D5) PPN:

$$\left\{ \bar{h}_0 \mid 0 \leq \bar{h}_0 \leq 1 \right\}$$

D6) OPN:

$$\left\{ \bar{h}_0 \mid 0 \leq \bar{h}_0 \leq \cos \theta_f \right\}$$

As capture area is concerned, we have $PPN = IPN \geq TPN \geq RTPN \geq GTPN$. The capture area of OPN is not fixed, depending on the weighting factor γ .

E) Time of Interception: τ_f .

E1) TPN:

$$\tau_f = \frac{\bar{h}_0}{R} \int_0^{\theta_f} \frac{e^{-\Phi_{TPN}(\theta)}}{\cos^2(\theta + \delta)} d\theta$$

E2) RTPN:

$$\tau_f = \bar{h}_0 \int_0^{\theta_f} \frac{e^{-\Phi_{RTPN}(\theta)}}{(\cos \psi_0 \cosh \sqrt{k}\theta - \sqrt{k} \sin \psi_0 \sinh \sqrt{k}\theta)^2} d\theta$$

E3) GTPN:

$$\tau_f = \bar{h}_0 \int_0^{\theta_f} \frac{e^{-\Phi_{GTPN}(\theta)}}{[k \cos(\theta + \delta) + \lambda \cos \psi_0]^2} d\theta$$

E4) IPN:

$$\tau_f = \bar{h}_0 \int_0^{\theta_f} \frac{e^{-\Phi_{IPN}(\theta)}}{\cos^2(k\theta + \psi_0)} d\theta$$

E5) PPN:

$$\tau_f = \bar{h}_0(\eta - 1)^2 \int_0^{\theta_f} \frac{e^{-\Phi_{PPN}(\theta)}}{[-\cos(\theta - \psi_0) + \eta \cos(k\theta + \psi_0)]^2} d\theta$$

E6) OPN:

$$\tau_f = \bar{h}_0 \theta_f^2 \int_0^{\theta_f} \frac{e^{-\Phi_{OPN}(\theta)}}{[\theta_f \cos(\theta - \psi_0) - \theta \cos(\theta - \theta_f) \cos(\theta_f - \psi_0)]^2} d\theta$$

where the first four characteristic functions $\Phi_{TPN}(\theta)$, $\Phi_{RTPN}(\theta)$, $\Phi_{GTPN}(\theta)$, and $\Phi_{IPN}(\theta)$ have been expressed in terms of the elementary functions of θ . Based on the above lists of performance characteristics, some relations among the various guidance laws can be revealed immediately.

Relation 1): $\lim_{\eta \rightarrow \infty} PPN(\eta) = IPN$. The first relation we have is that when the interceptor has an absolute speed superiority over the target, then PPN and IPN become indistinguishable. The performance of PPN depends on the value of η which is the speed ratio of the interceptor over the target. We denote this performance dependence by $PPN(\eta)$. If we let $\eta \rightarrow \infty$, all the PPN formula listed above converge to those of IPN. Mathematically speaking, we have $\lim_{\eta \rightarrow \infty} PPN(\eta) = IPN$. Although IPN may not be of practical use, the study of IPN can, at least, tell us the limits of the performance of PPN.

Relation 2): $TPN = RTPN = GTPN = IPN$, as $\psi_0 \rightarrow \pi/2$. The second relation says that under the tail-chase situation, the four guidance laws, namely, TPN, RTPN, GTPN, and IPN, become identical. It is noted that when $\psi_0 = \pi/2$, $\bar{h}_0 = \cos(\pi/2) = 0$. From the definition of $\bar{h}_0$, $\bar{h}_0 = 0$ represents the situation where the target escapes initially along the LOS. From the performance lists A-E, we can observe that when $\psi_0 \rightarrow \pi/2$, all the formula of TPN, RTPN, GTPN, and IPN become identical. It is also noted that $\theta(t) = \theta_f = 0$, $t \geq 0$, for the four guidance laws, if $\psi_0 = \pi/2$. This is simply the case that the interceptor follows a straight-line course during the interception for a nonmaneuvering target under tail-chase condition.

Relation 3): $\lim_{\theta \rightarrow \theta_f} OPN = PPN$. The third relation says that toward the end of interception, the behavior of OPN and PPN becomes closer and closer. The behavior of PPN depends on the speed ratio η , while the behavior of OPN depends on the weighting factor γ . It is possible for certain choice of η and γ that the behavior of OPN and PPN can be similar. From (42) and (58), the interceptor's acceleration according to PPN and OPN can be described, respectively, as

$$(\bar{a}_I)_{PPN} = \lambda V_I \dot{\theta} [-\sin(k\theta + \alpha_0) \bar{e}_r + \cos(k\theta + \alpha_0) \bar{e}_\theta] \quad (63a)$$

$$(\bar{a}_I)_{OPN} = \frac{\lambda^2 A}{2\gamma} \dot{\theta} [-\sin(-\theta + \theta_f) \bar{e}_r + \cos(\theta - \theta_f) \bar{e}_\theta]. \quad (63b)$$

Under certain design conditions such as $\gamma = \lambda A / (2V_I)$, $\alpha_0 = -\theta_f$, $k = \lambda - 1 = 1$, we have $\lim_{\theta \rightarrow \theta_f} (\bar{a}_I)_{PPN} = \lim_{\theta \rightarrow \theta_f} (\bar{a}_I)_{OPN} = \lambda V_I \dot{\theta} \bar{e}_\theta$. The interpretation of this result is that PPN is an optimal guidance law from the

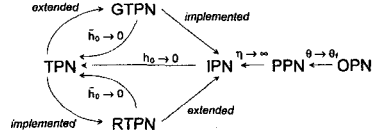


Fig. 2. Relations among 6 guidance laws.

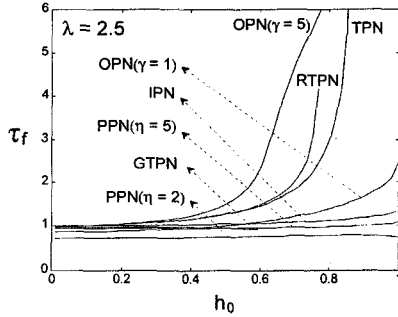


Fig. 3. Time of interception with $\lambda = 2.5$.

sense of the optimization of the performance index I in (50) with proper choice of weighting factor γ . This fact may somewhat explain the superior performance of PPN. Fig. 2 summarizes the relations among the six guidance laws.

V. NUMERICAL RESULTS

Using the mathematical formula derived previously, the six guidance laws can be compared on the same basis, and more physical insights can be gained from numerical studies. The influence of parameters λ , ψ_0 , η , and γ , are discussed separately.

A. The Effect of λ

Two cases with $\lambda = 2.5$ and $\lambda = 6$ are considered here. It is observed that when λ is changed from 2.5 to 6, the required time of interception τ_f is reduced for every guidance law (see Figs. 3 and 4), and at the same time, the cumulative velocity increment of every guidance law is also decreased (see Figs. 5 and 6). As expected, the navigation constant λ has a similar effect on each PN scheme, and the performance gets better with an increasing λ . A λ between 3 and 6 would be a good choice for global performance. A λ greater than 6 is not recommended to avoid instability caused by time lags and sensor noises.

B. The Effect of ψ_0 ($\bar{h}_0$)

The variable ψ_0 is defined as $\bar{h}_0 = \cos \psi_0 = r_0 \dot{\theta}_0 / \sqrt{\dot{r}_0^2 + r_0^2 \dot{\theta}_0^2}$. The value of $\bar{h}_0$ reflects the tendency of the target escaping in the tangential direction. Hence, $\psi_0 = \pi/2$ ($\bar{h}_0 = 0$) stands for the tail-chase condition where the target escapes along the radial direction (LOS), while $\psi_0 = 0$ ($\bar{h}_0 = 1$) corresponds

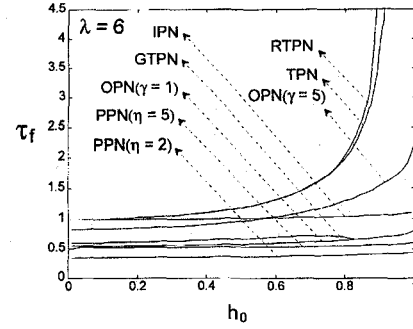


Fig. 4. Time of interception with $\lambda = 6$.

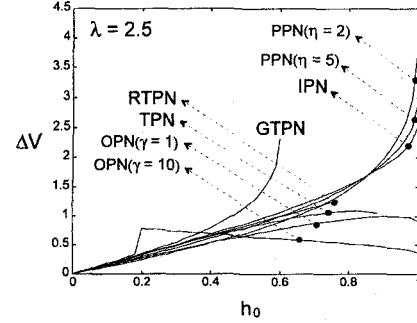


Fig. 5. Cumulative velocity increment with $\lambda = 2.5$.

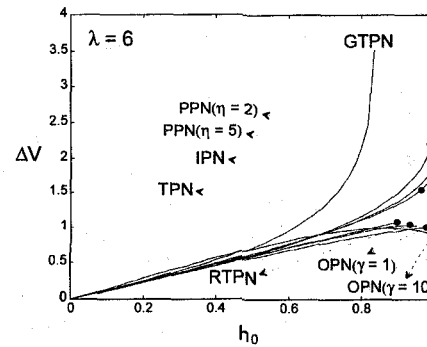


Fig. 6. Cumulative velocity increment with $\lambda = 6.0$.

to the worst-case scenario where the target escapes purely in the tangential direction. When the target's motion is near the tail-chase condition (i.e., small $\bar{h}_0$), there is no significant difference in the behavior of the six PN schemes as can be seen from Figs. 3 and 5. However, when $\bar{h}_0$ is increased toward its maximum allowable value, drastic degradation of performance is observed in most PN schemes. The maximum allowable $\bar{h}_0$ for each PN scheme is determined by the capture conditions listed in property D1–D6.

According to the performance at $(\bar{h}_0)_{\max}$, the six PN schemes can be divided into two groups. The first group consists of TPN, RTPN, and OPN with large γ . The required time of interception in this group exhibits an abrupt ascent when $\bar{h}_0$ is increased toward $(\bar{h}_0)_{\max}$ as shown in Figs. 3 and 4. However, the associated increase in the cumulative velocity increment ΔV is relatively small in this group, as

can be seen from Figs. 5 and 6. The second group which possesses the reverse properties of group 1 comprises PPN, IPN, GTPN, and OPN with small γ . In second group, the increase in the required time of interception is insignificant even near the worst-case condition ($(\bar{h}_0)_{\max} \rightarrow 1$); however, the cumulative velocity increment grows precipitously (see Figs. 5 and 6). Among the six PN schemes, OPN is the only one that is able to compromise the required time of interception with the required energy consumption by proper choice of the weighting factor γ . An OPN with large γ puts main concern on energy consumption, and the resulting performance is similar to TPN (group 1); while an OPN with small γ put major penalty on the time of interception, and the resulting performance is close to PPN (group 2). The OPN with $\gamma = 1$ having equal weights on both requirements seems to be a good compromise (see Figs. 3 and 5).

C. The Effect of η

The variable η is the speed ratio of the interceptor over the target. Two speed ratios $\eta = 2$ and $\eta = 5$ are considered in the numerical comparison. It is found that the PPN with $\eta = 2$ demonstrates shorter interception time than that of PPN with $\eta = 5$ (see Figs. 3 and 4), but requires more energy consumption (see Figs. 5 and 6). As has been proved in the last section, IPN is the extreme case of PPN as $\eta \rightarrow \infty$. Hence, we can see from Figs. 3 and 4 that the τ_f curve of IPN constitutes an upper envelop of the curves of PPN with various speed ratios, and from Figs. 5 and 6 we can observe that the ΔV curve of IPN constitutes a lower envelop of the various PPN curves. Because the distinction between the PPN with $\eta = 2$ and the PPN with $\eta = \infty$ (IPN) is not apparent, a PPN scheme employing large speed ratio seems impractical and unnecessary.

D. The Effect of γ

The weighting factor γ has a remarkable influence on OPN. The numerical results show that the other five PN schemes can be roughly approximated by OPN with certain choice of γ . Generally speaking, the behavior of OPN with small γ is close to the behavior of PPN, while the OPN with large γ is close to TPN. Fig. 7 shows that the $\bar{h}(\theta)$ trajectory of OPN with $\gamma = 1$ almost coincides with that of PPN with $\eta = 5$, and Fig. 8 shows that the trajectory of OPN with $\gamma = 8.5$ is close to that of TPN. Figs. 7 and 8 also manifest the difference in the convergent speed of the relative motion for different $\bar{h}_0$. In Fig. 7 $\bar{h}_0$ is set to 0.5 for all six guidance laws, and in Fig. 8 $\bar{h}_0$ is set to its maximum allowable value that corresponds to the worst-case condition of each guidance law. It can be observed that under the worst-case condition, the trajectories of $\bar{h}(\theta)$ are more sluggish than the

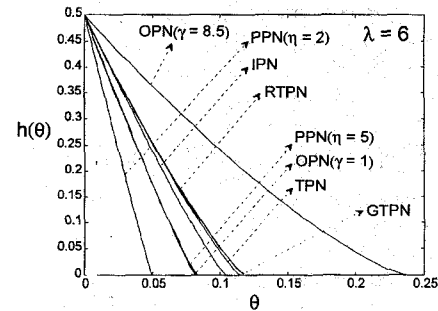


Fig. 7. Trajectories of $\bar{h}(\theta)$ with $\bar{h}_0 = 0.5$.

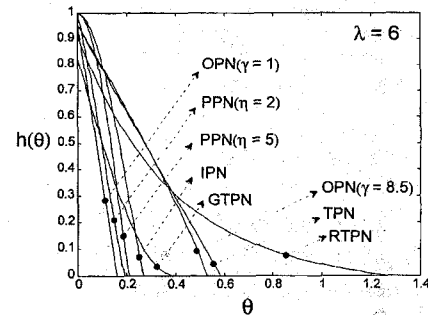


Fig. 8. Trajectories of $\bar{h}(\theta)$ with maximum allowable $\bar{h}_0$.

case of $\bar{h}_0 = 0.5$, indicating that the convergence of the relative motion is more slow.

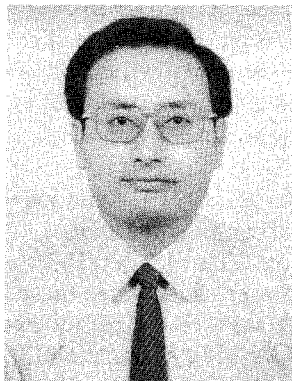
VI. CONCLUSIONS

An attempt has been made to unify the mathematical tools used in analyzing the existing PN guidance laws. With the introduction of a generalized PN scheme, the six guidance laws, namely, TPN, RTPN, GTPN, IPN, PPN, and OPN, are included as its special cases, and are solved in a unified manner. Along this approach, the experiences gained from the analytical study of TPN-type guidance laws can be equally applied to the PPN-type guidance laws. The concept proposed in this paper may hopefully pave the way for combining the advantages of TPN in mathematical tractability with the advantages of PPN in practical implementation.

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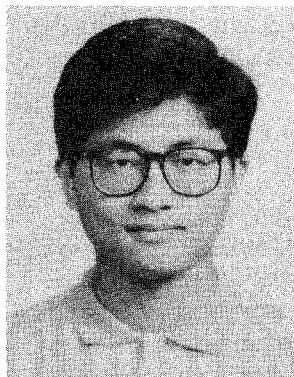
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Ciann-Dong Yang received his B.S., M.S., and Ph.D. degrees in aeronautics and astronautics from National Cheng Kung University, Taiwan, in 1983, 1985, and 1987, respectively.

Since 1985, he has been with the Department of Aeronautics and Astronautics, National Cheng Kung University, where he is currently a full Professor. His research interests are in H_∞ robust control and missile guidance.

Dr. Yang is the recipient of the 1990 M. Barry Carlton Award and the Outstanding Paper Award of the IEEE Aerospace and Electronic Systems. He is also the author of *Engineering Analysis* and is the coauthor of *Post Modern Control Theory and Design*.



Chi-Ching Yang was born in Lukang, Taiwan, on Aug. 18, 1969. He received his B.S. degree in mechanical engineering from Chung Yuan University, Taiwan, in 1991 and the M.S. and Ph.D. degrees in aeronautics and astronautics from National Cheng Kung University, Taiwan, in 1993 and 1996.

He is currently a postdoctorate researcher in the Institute of Aeronautics and Astronautics, National Cheng Kung University, Taiwan. His research interests are mainly in missile guidance and control theory.